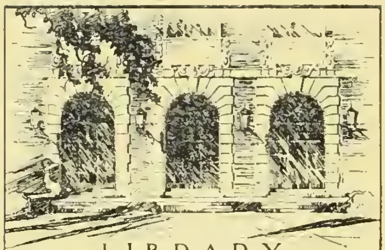

HIGH SCHOOL MATHEMATICS

Unit 6.

GEOMETRY

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

MAX BEBERMAN, *Director*

HERBERT E. VAUGHAN, *Editor*

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A VISIT TO THE PLANET GLOX

After many years of experimenting, the missile scientists finally sent a space ship to the planet Glox. Of course, the ship was small and had just one man as crew. His job was to explore the planet, and since the job was a dangerous one, he was instructed to radio information back to the home base as soon as he could. Establishing radio contact was very difficult, so he was also told to send back only important information.

As the space man approached Glox, he noticed a city. So he sent back his first message:

> <u>Message 1.</u>
>
> Glox has at least one city.

As he moved closer, he noticed a highway, and sent back:

> <u>Message 2.</u>
>
> Glox has at least one highway.

His commander radioed back and told him to find out more about the highway network.

The spaceman landed and began to explore. He found three cities, and discovered that no single highway passed through all three of them. Feeling that he should report this, he returned to the ship and sent:

> <u>Message 3.</u>
>
> Glox has at least three cities not all on the same highway.

The next day he met a Gloxian. The Gloxian seemed harmless, so the space man tried to talk to him. Since the Gloxian did not seem to understand English, the space man tried sign language. After great effort, he managed to learn that each highway went through at least two cities. Back to the radio!

Message 4.

Each highway on Glox passes through at least two cities.

Another day of "talking" with the Gloxian yielded still more information:

Message 5.

You can travel from any city on Glox to any other without changing highways. Moreover, there is only one such highway connecting each pair of cities.

During the next day, the space man was visited by a crowd of Gloxians who were curious about the ship. The public-relations-minded space man decided to allow each Gloxian in the crowd to sit in the pilot's seat. One of the Gloxians, in an exploratory mood, pushed the wrong combination of buttons and severely damaged the power supply for the radio. After the crowd had departed, the space man discovered the damage and estimated that he would need a long time to repair it.

When the commander failed to hear from the explorer after an uncomfortably long period, he decided to send a second ship out to Glox. The second space man was also instructed to send back important information about the highway network, and to try to find the first space man. Before leaving, he was given copies of the first five messages so that he would not waste time and power in transmitting the same information. As the second space man came close to Glox, he saw cities and highways, but since these facts had already been reported, he did not send back this information.

Upon landing, he decided to explore. During the course of several days of hard walking along highways and across fields, he found

highways marked by signs like these:

Guessing that these were highway signs like those of his own country, he concluded that Glox had at least three highways. So, he sent:

> Message A.
>
> Glox has at least three highways.

The next day a Gloxian told him the following fact:

> Message B.
>
> No two highways meet in more than one city.

After another day of laborious sign language, he reported:

> Message C.
>
> No single highway goes through all cities on Glox.

The next day the commander radioed back:

> FIRST MAN IS SAFE. YOU COME HOME AT ONCE. YOU'RE WASTING TIME!

When he returned, the commander fired him, and explained,

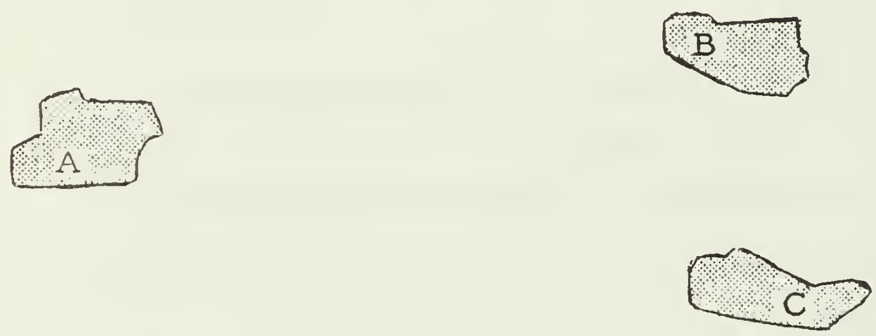
"Your messages were completely unnecessary. After reading the other man's messages, my daughter, Jo, who is only a sophomore at Zabranchburg High School, gave me the information you sent in Messages A, B, and C long before you sent them!"

* * *

Do you see how Jo did it?

Here is how Jo found out what Message A said, that is, that Glox had at least three highways.

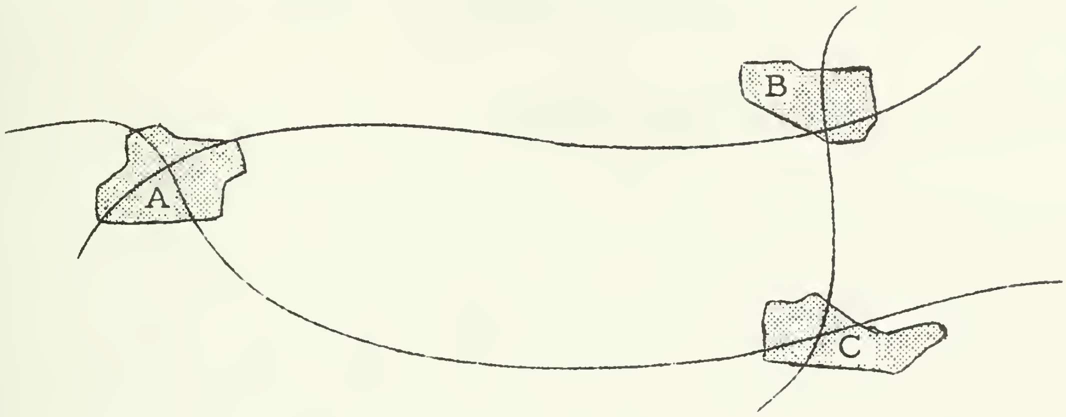
Message 3 told her that Glox had at least three cities not all on the same highway. She decided to think of three such cities as city A, city B, and city C. To make it easier for her to think about these cities, she represented them by spots on a sheet of paper:



Message 5 told her that there was one and only one highway connecting each pair of cities. So, she concluded, Glox had a highway connecting city A and city B -- the AB-highway. She represented this highway by drawing a line on her diagram, like this:

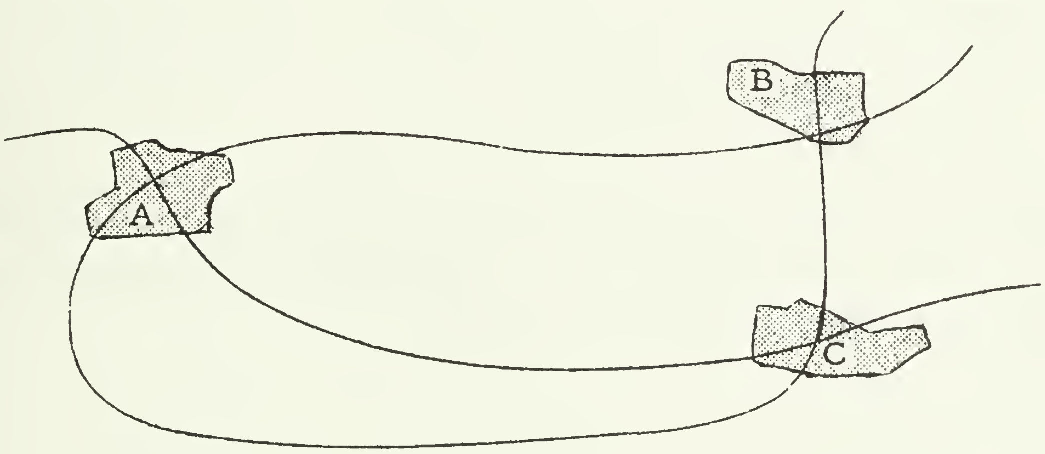
Similarly, she reasoned that there was a highway connecting B and C --

the BC-highway; also, there was a highway connecting C and A -- the CA-highway. Her diagram then looked like this:



From her diagram it seemed clear to her that Glox had at least three highways.

But, then she stopped, and wondered: Maybe these highways are not all different. Maybe the AB-highway is the same as the BC-highway. Maybe my picture should look like this:



Then she recalled that cities A, B, and C were not on the same highway. So, C could not be on the AB-highway. Since C was on the BC-highway, it was clear that the AB-highway and the BC-highway were different. In the same way, she reasoned that the CA-highway was different from both theAB-highway and the BC-highway. So, the AB-highway, the BC-highway, and the CA-highway were all different. This meant that Glox had at least three highways.

Try to write a paragraph showing the reasoning Jo might have used to deduce Message B. You can draw a picture to help you visualize the situation, but you must not try to make the picture part of the proof.

Then try your hand on Message C.

* * *

In Zabranchburg, a group of businessmen formed partnerships among themselves.

(1) At least three of the businessmen did not belong to the same partnership.

(2) Each partnership consisted of at least two businessmen.

(3) For each two businessmen, there was one and only one partnership which contained both of them.

Do you think it is possible to deduce from just these three facts that there are at least three partnerships among these businessmen? Can other things be deduced from these facts?

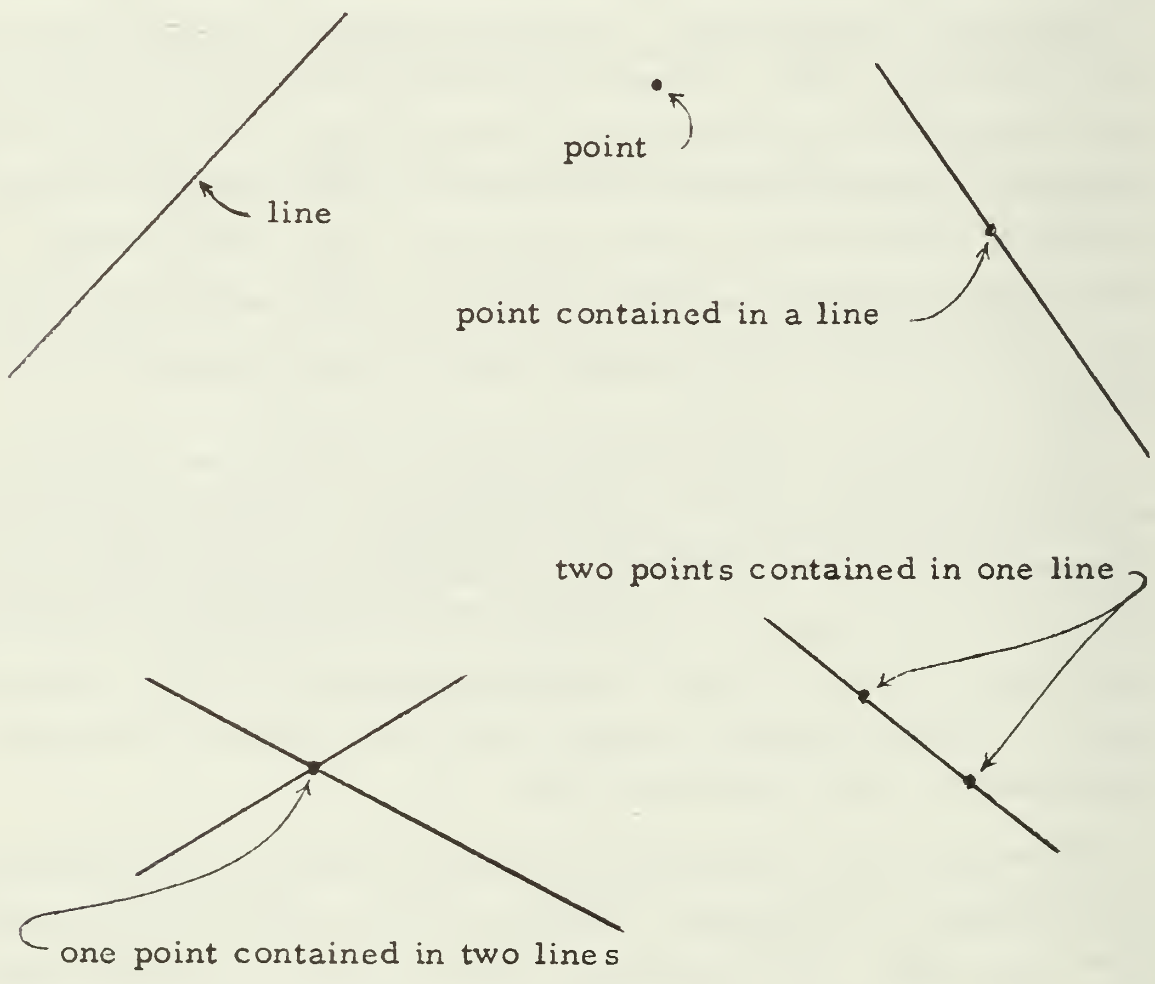
GEOMETRY

In this unit we shall study geometry in a systematic manner. The study of geometry grew out of people's need to measure land and out of their later interest in all kinds of spatial relationships. Much had been discovered about these relationships by the time that the Greek mathemetician Euclid [300 B.C.] first arranged the discoveries in a systematic way. His work, contained in thirteen books [his famous Elements], summarized the work of a number of his predecessors and contemporaries. These people observed certain forms occurring over and over again in nature and in man-made structures--forms like stretched ropes, the straight edges of blocks, and the straight sides of tall trees. From many observations of physical objects, they arrived at the idea of an "abstract" straight line. In similar fashion, they arrived at the idea of an "abstract" point. These ideas had probably been arrived at by other peoples also, but they did not discover how to use them to full advantage. This the Greeks did discover. They discovered that it was possible to reason from certain information about lines, points, and other abstract objects, and thus obtain new information about these objects. This discovery has turned out to be one of the greatest ever made by man.

The process of using reasoning alone to obtain new information from information already known is called deduction. Euclid applied deduction to geometry in a systematic way. He set down certain statements containing basic information about abstract lines and points. From these statements, he deduced other statements [just as Jo did], and the statements he deduced seemed to apply to physical objects to the same extent that his basic statements did.

Nowadays, basic statements of the kind Euclid started with are called axioms or postulates, and the statements deduced from them are called theorems. Euclid probably did not realize, as Jo did, that it didn't really matter what objects his axioms talked about as long as these objects had the properties expressed by the axioms. A great deal of mathematical progress today depends upon the clear recognition of this fact. [Compare the businessmen-partnerships problem with the cities-highways problem.]

In studying geometry, we shall state certain axioms about points and lines from which we can deduce theorems. As far as deduction is concerned, it doesn't really matter what points and lines are, but since our axioms will--to the same extent as Euclid's did-- express spatial relationships, it may help you, especially in discovering theorems, to think of points and lines as objects abstracted from the physical world. For this reason, it will be helpful to picture points and lines by drawing tiny dots and streaks. [Of course, you should think of the streaks as going on forever in each direction.] However, regardless of what the pictures may suggest, we shall assume only that lines and points are objects that satisfy the axioms.



In this Introduction, we shall do a lot of exploring by drawing pictures in order to discover relationships among points and lines in a plane. Our purpose now is to collect information rather than to systematize it deductively. At the end of the Introduction we shall summarize the information we have obtained so far.

POINTS AND LINES IN A PLANE

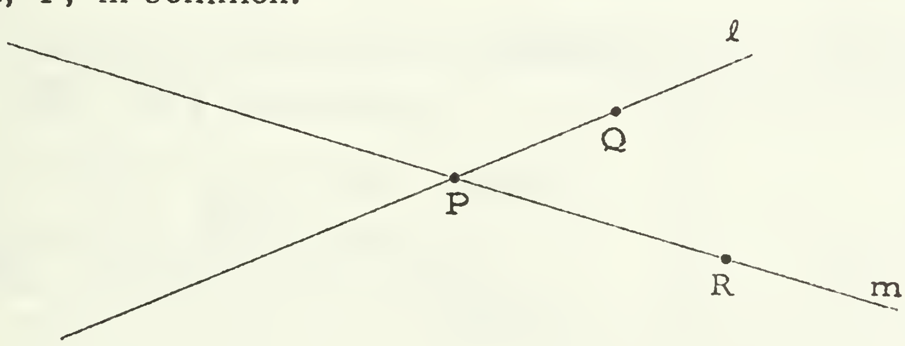
When you draw pictures of geometric figures or look at such pictures, you may think of the geometric figures as sets of points. All the geometric figures we shall deal with will be subsets of a plane. You can imagine a plane to be a flat surface that extends forever in all directions.

Among the simplest geometric figures are straight lines. You can draw a picture of part of a straight line with a ruler. [Since a straight line goes on forever in both directions, you can't draw a picture of all of a straight line.] A straight line is a set of points. If you mark a point on a piece of paper, you can draw pictures of as many straight lines as you like which contain this point.

If you mark two points on a piece of paper, how many straight lines can you picture which contain both of these points?

Two straight lines can have a point in common. Can two straight lines have more than one point in common? Can two straight lines have less than one point in common?

Here is a picture of two straight lines, ℓ and m, which have one point, P, in common.



$\ell \cap m = \{P\}$

If $Q \in \ell$ [that is, if Q belongs to ℓ, or is a point on ℓ] and $Q \neq P$ then we say that ℓ is the line which contains P and Q [or is the line determined by P and Q].

In the case pictured above we say that ℓ is $\overleftrightarrow{PQ}$. [Read '$\overleftrightarrow{PQ}$' as 'the line PQ'.] Refer to the picture, and tell what $\overleftrightarrow{PR}$ is. What is $\overleftrightarrow{PQ} \cap \overleftrightarrow{PR}$? What is $\overleftrightarrow{PQ} \cap \overleftrightarrow{RQ}$?

Mark two points, A and B. Draw a picture of $\overleftrightarrow{AB}$. Now, draw a picture of $\overleftrightarrow{BA}$. Is it the case that $\overleftrightarrow{AB} = \overleftrightarrow{BA}$? What is $\overleftrightarrow{AB} \cap \overleftrightarrow{BA}$?

Mark a point C such that $C \notin \overleftrightarrow{AB}$. Is there a straight line which contains all three points A, B, and C? Draw a line ℓ such that $C \in \ell$ and $\ell \cap \overleftrightarrow{AB} = \emptyset$. How many such lines are there?

Two lines whose intersection is empty [that is, which have no common point] are said to be parallel lines. So, if s is a line and t is a line then s is parallel to t if and only if $s \cap t = \emptyset$.

Is a line parallel to itself? If s is parallel to t, does it follow that t is parallel to s? If ℓ is parallel to m and m is parallel to n, does it follow that ℓ and n are parallel?

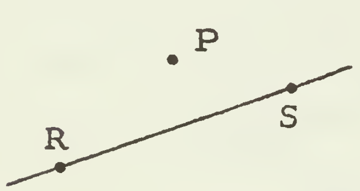
Mark three points, M, N, and R, all of which belong to the same line. Is it the case that $\overleftrightarrow{MN} = \overleftrightarrow{NR}$? M, N, and R are said to be collinear points. A set of points is a set of collinear points if and only if they all belong to the same line.

EXERCISES

A. Draw pictures which illustrate the situations described below. If the situation can't occur, tell why.

Sample 1. $P \notin \overleftrightarrow{RS}$

Solution.



Sample 2. A, B, and C are three points and $C \notin \overleftrightarrow{AC} \cap \overleftrightarrow{BC}$

Solution. Can't occur. Since $C \in \overleftrightarrow{AC}$ and $C \in \overleftrightarrow{BC}$, it follows that C belongs to the intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$.

1. $A \neq B$ and $C \in \overleftrightarrow{AB}$
2. ℓ and m are two lines and $P \in \ell \cap m$
3. ℓ and m are two lines, $P \neq R$, and $\{P, R\} \subseteq \ell \cap m$
4. A, B, C, and D are four points, $A \in \overleftrightarrow{BC}$, and $B \in \overleftrightarrow{AD}$
5. r and s are two lines and $r \cap s \neq \emptyset$
6. r and s are two lines and $r \cap s = \emptyset$
7. r, s, and t are three lines and $\{P\} = (r \cap s) \cap t$

8. r, s, and t are three lines and $P \in (r \cap s) \cap t$

9. r, s, and t are three lines, $P \neq Q$, and $\{P, Q\} \subseteq (r \cap s) \cap t$

10. r, s, and t are three lines, $P \neq Q$, and $\{P, Q\} = t \cap (r \cup s)$

11. A, B, and C are three collinear points and D is a fourth point such that $D \in \overleftrightarrow{BC}$

12. A, B, and C are three collinear points, B, D, and E are three collinear points, and $D \notin \overleftrightarrow{AB}$

13. M, N, and R are three collinear points, and M, N, and S are three <u>noncollinear</u> points

14. A, B, and C are three collinear points, B, C, and D are three noncollinear points, and A, C, and D are three collinear points

15. A, B, C, and D are four noncollinear points, $B \in \overleftrightarrow{AC}$, and $B \in \overleftrightarrow{AD}$

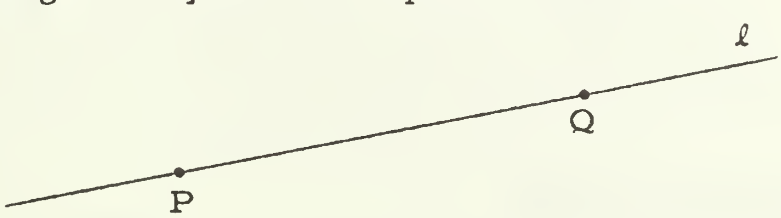
☆<u>B.</u> 1. Suppose s is a set of points each three of which are collinear. Is s a set of collinear points?

2. Suppose r is a set of points each two of which are collinear; does it follow that r is a set of collinear points?

3. Suppose t is a set of collinear points such that, for each point P, if $P \notin t$ then $t \cup \{P\}$ is not a set of collinear points. What kind of set is t?

ORDER OF POINTS ON A LINE

Here is a picture of a line ℓ [from now on, 'line' means the same thing as 'straight line'] and of two points P and Q on ℓ.



Mark another point, V, on ℓ. Now study the statements given below, and underline those which appear to be true.

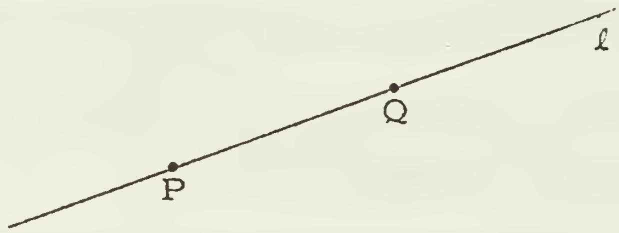
(1) Q is between P and V
(2) Q is between V and P
(3) V is between P and Q
(4) V is between Q and P
(5) P is between Q and V
(6) P is between V and Q

If A, B, and C are three points and one of these points is between the other two then the three points are collinear. If D, E, and F are three noncollinear points then none of them is between the other two.

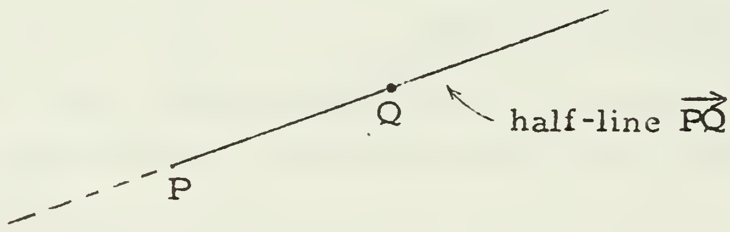
Draw three collinear points. Of how many of these points can you say that it is between the other two?

Now, draw a line and mark on it four points, A, B, C, and D, such that B is between A and C, and C is between B and D. Is B between A and D? Is C between A and D?

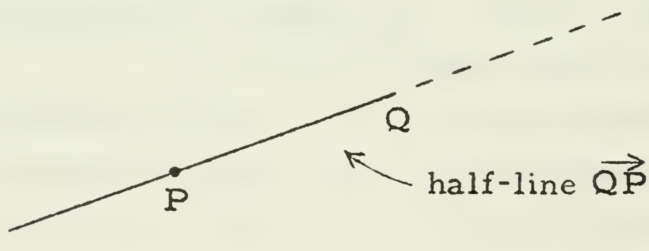
Consider, again, the line ℓ and the points P and Q. The set consisting of the points of ℓ which are on the same side of P as is Q is called



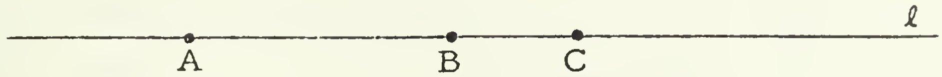
<ins>the half-line from P through Q</ins>. It is designated by the symbol '$\overrightarrow{PQ}$'. Here is a picture of $\overrightarrow{PQ}$:



and here is a picture of $\overrightarrow{QP}$:



Consider the points A, B, and C on ℓ where B is between A and C.



Suppose D $\in \ell$. From which of the following conditions does it follow that D $\in \overrightarrow{BC}$?

(a) D is between A and C

(b) D is between B and C

(c) A is between B and D

(d) C is between B and D

(e) D = C

(f) B is between A and D

What is the set of all points X such that B is between A and X? What is $\{X$: B is between C and X$\}$?

Suppose M and N are two points of line k. [Draw a picture.] Is $\overrightarrow{MN} = \overrightarrow{NM}$? What set of points is $\overrightarrow{MN} \cup \overrightarrow{NM}$? Describe the set of points $\overrightarrow{MN} \cap \overrightarrow{NM}$.

The set consisting of those points of k which are between M and N is called <u>the interval joining M and N</u>, and is designated by the symbol '$\overline{MN}$'. So, $\overline{MN} = \overrightarrow{MN} \cap \overrightarrow{NM}$. Is $\overline{MN} = \overline{NM}$?

Consider the two points A and B on ℓ.

Suppose C $\in \ell$. From which of the following conditions does it follow that C $\in \overline{AB}$?

(a) C is between A and B

(b) C is between B and A

(c) C $\in \overrightarrow{AB}$

(d) C $\in \overrightarrow{BA}$

(e) C $\in \overrightarrow{AB}$ and C $\in \{X$: B is between A and X$\}$

(f) C $\in \overrightarrow{AB}$ and C $\neq$ B and C $\notin \{X$: B is between A and X$\}$

(g) C $\in \{X$: X is between A and B$\}$

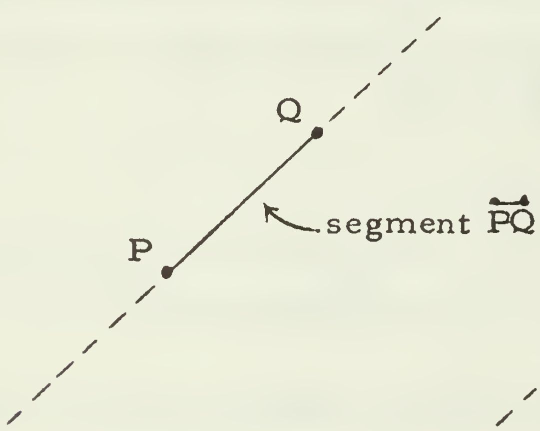
Mark four points, A, B, C, and D, such that B $\in \overline{AC} \cap \overline{AD}$. [Must the four points be collinear? Why?] Does C $\in \overline{BD}$? Does D $\in \overline{BC}$? Given four such points, does it follow that the third is between the second and the fourth? That either the third is between the second and the fourth or the fourth is between the second and the third?

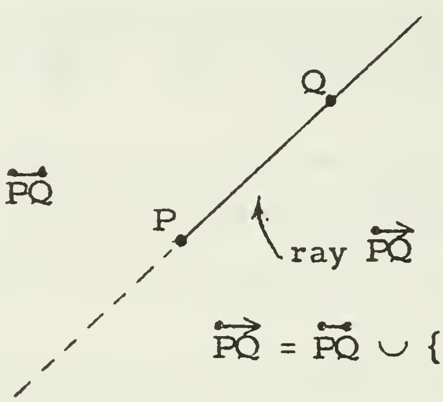
The half-line $\overrightarrow{PQ}$ is the set of all points on the line $\overleftrightarrow{PQ}$ which are on the Q-side of P. Thus, the point P, itself, does not belong to $\overrightarrow{PQ}$. Similarly, the point P does not belong to the interval $\overline{PQ}$. The points, P and Q, are said to be the <u>end points</u> of the interval $\overline{PQ}$. Also, the point P is said to be the <u>vertex</u> of the half-line $\overrightarrow{PQ}$. So, the end points of an interval are the vertices of the two half-lines which determine the interval.

We shall sometimes have occasion to talk about the set $\{P\} \cup \overrightarrow{PQ}$ and the set $\{P, Q\} \cup \overline{PQ}$.

$\{P\} \cup \overrightarrow{PQ}$ is called <u>the ray from P through Q</u> [or: the closed half-line from P through Q]. The ray from P through Q is designated by the symbol '$\overset{\bullet\!\rightarrow}{PQ}$'.

$\{P, Q\} \cup \overline{PQ}$ is called <u>the segment joining P and Q</u> [or: the closed interval between P and Q]. This geometric figure is designated by the symbol '$\overset{\bullet\!-\!\bullet}{PQ}$'.





$$\overset{\bullet\!\rightarrow}{PQ} = \overset{\bullet\!-\!\bullet}{PQ} \cup \{Z :\ Q \in \overline{PZ}\}$$

Briefly, a ray contains its vertex, while a half-line does not, and a segment contains its end points, while an interval does not.

EXERCISES

Draw pictures which illustrate the situations described. If a situation can't occur, tell why.

<u>Sample 1.</u> $P \in \overline{AB}$

<u>Solution.</u>

<u>Sample 2.</u> $P \in \overleftrightarrow{AB}$ but $P \notin \overset{\bullet\!\rightarrow}{AB}$

<u>Solution.</u>

1. A, B, and C are collinear

2. $P \in \overline{MN}$ and Q is between P and N

3. $\overline{AB} \subseteq \overline{CD}$

4. $\overline{BC} \subseteq \overrightarrow{AB}$

5. A, B, and C are collinear and $\overrightarrow{AB} \not\subseteq \overset{\bullet\!\!\rightarrow}{AC}$

6. $K \in \overline{AB}$ and $B \in \overline{AK}$

7. $\overline{MN} \cap \overline{AB} = \{P\}$

8. $P \in \overline{AB}$, $P \in \overline{DC}$, and $P \in \overline{AD}$

9. $\overrightarrow{AP}$, $\overrightarrow{AQ}$, and $\overrightarrow{AR}$ are three half-lines, and $\overline{PQ} \cap \overrightarrow{AR} \neq \emptyset$

10. $\overrightarrow{AP}$, $\overrightarrow{AQ}$, and $\overrightarrow{AR}$ are three half-lines and $\overline{PQ} \cap \overrightarrow{AR} = \emptyset$

11. P, Q, and R are noncollinear points and $Q \in \overline{PR}$

12. $\{R, S\} \neq \{A, B\}$ and $\overleftrightarrow{AB} = \overleftrightarrow{RS}$

13. $B \in \overleftrightarrow{AC}$, $B \notin \overrightarrow{AC}$, and $B \notin \overrightarrow{CA}$

14. $\overleftrightarrow{AB} = \overrightarrow{AC} \cup \overrightarrow{AB}$

15. $\overleftrightarrow{AB} = \overset{\bullet\!\!\rightarrow}{AC} \cup \overset{\bullet\!\!\rightarrow}{AB}$

16. $\overrightarrow{AB} = \overline{AB} \cup \overset{\bullet\!\!\rightarrow}{BC}$

17. $\overrightarrow{AB} = \overset{-\!\bullet}{AB} \cup \overrightarrow{BC}$

18. $\ell \parallel m$ and $m \parallel n$ [that is, ℓ is parallel to m and m is parallel to n]

19. $\ell \parallel m$, $m \parallel n$, and $\ell \nparallel n$

20. $\ell \parallel m$ and ℓ intersects $\overleftrightarrow{AB}$ in a point between A and B

21. $\overleftrightarrow{AB} \neq \overleftrightarrow{AC}$ and $B \in \overleftrightarrow{AB} \cap \overleftrightarrow{AC}$

22. $C \in \{Z\colon$ Z is between A and B$\}$

23. $C \neq A$, $C \neq B$, and $C \in \overline{AB} \cup \{A, B\}$

24. $C \neq A$, $C \neq B$, and $C \in \overset{\bullet\!-\!\bullet}{AB} \cup \{Z\colon B \in \overline{AZ}\}$

25. $A \in \{Z\colon B \in \overline{AZ}\}$

26. $\{Z\colon B \in \overline{AZ}\} = \emptyset$

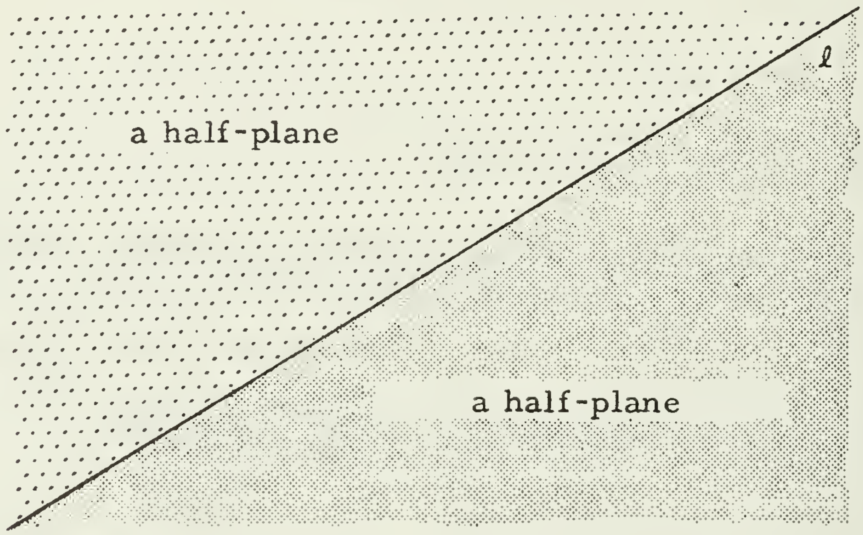
27. $B \in \overline{AC} \cap \overline{AD}$, $C \notin \overline{BD}$, and $C \neq D$

28. A, B, and C are three collinear points, $A \notin \overline{BC}$, and $C \notin \overline{AB}$

29. $B \in \overline{AC}$, $D \in \overrightarrow{BC}$, and $B \notin \overline{AD}$

SEPARATION OF THE PLANE

Just as a point separates a line into two half-lines, so the complement of a line [with respect to the plane] is the union of two <u>half-planes</u>. Each half-plane is said to be a <u>side</u> of the line ℓ, and the line ℓ is said to



be the <u>edge</u> of each half-plane. A half-plane contains no points of its edge.

Suppose h_1 and h_2 are the two sides of line ℓ. What is $h_1 \cup \ell \cup h_2$? What is $h_1 \cap \ell$? What is $h_1 \cap h_2$?

Suppose that $\overleftrightarrow{AB} \cap \ell = \emptyset$. Does it follow that A and B are on the same side of ℓ? Suppose that $A \in h_1$ and $C \in h_1$. Does it follow that $\overleftrightarrow{AC} \subseteq h_1$?

Suppose that $\overleftrightarrow{AB} \cap \ell = \emptyset$ and $\overleftrightarrow{BC} \cap \ell = \emptyset$. Does it follow that $\overleftrightarrow{AC} \cap \ell = \emptyset$?

Suppose that $\overleftrightarrow{AB} \cap \ell = \emptyset$ and $\overleftrightarrow{BC} \cap \ell \neq \emptyset$. Does it follow that $\overleftrightarrow{AC} \cap \ell \neq \emptyset$?

Suppose that $\overleftrightarrow{PQ} \cap \ell \neq \emptyset$. Does it follow that P and Q are on opposite sides of ℓ?

Suppose that $\overline{PQ} \cap \ell \neq \emptyset$. Does it follow that P and Q are on opposite sides of ℓ?

Suppose that $P \notin h_2$ and $Q \in h_1$. Does it follow that $\overline{PQ} \subseteq h_1$? That $\overrightarrow{PQ} \subseteq h_1$?

If $P \in \ell$ and $Q \in h_1$, does it follow that $\overrightarrow{PQ} \subseteq h_1$?

EXERCISES

Draw pictures which illustrate the situations described below. If a situation can't occur, tell why.

1. ℓ is a line which separates the plane into the two half-planes, h_1 and h_2, $\overleftrightarrow{AB} \parallel \ell$, $A \in h_1$, and $B \in h_2$

2. ℓ determines the two half-planes, h_1 and h_2, m determines the two half-planes, k_1 and k_2, and $h_1 \subseteq k_1$

3. $\overleftrightarrow{AB} \cap \ell = \emptyset$, $\overleftrightarrow{BC} \cap \ell = \emptyset$, and $\overleftrightarrow{AC} \cap \ell \neq \emptyset$

4. $M \in \overline{AB}$, $N \in \overline{AC}$, A, B, and C are three noncollinear points, and $\overleftrightarrow{MN}$ is a subset of that side of $\overleftrightarrow{BC}$ that contains A

5. $M \in \ell$, $N \notin \ell$, and $\overrightarrow{MN}$ is a subset of that side of ℓ that contains N

6. $M \in \ell$, $N \notin \ell$, and $\overrightarrow{NM}$ is a subset of that side of ℓ that contains N

*

A half-plane together with its edge is called <u>a closed half-plane</u>. Suppose ℓ determines the two closed half-planes k_1 and k_2. What is $k_1 \cup k_2$? What is $k_1 \cap k_2$?

*

7. ℓ determines the two closed half-planes k_1 and k_2, $A \in k_1$, $B \in k_2$, and $\overline{AB}$ does not cross ℓ [$\overline{AB}$ <u>crosses</u> ℓ if and only if $\overline{AB}$ contains points of both sides of ℓ.]

8. c is a closed half-plane determined by line m, k is a closed half-plane determined by line n, m and n cross each other, $P \in c \cap k$, and $P \notin m \cup n$

9. k is a closed half-plane determined by ℓ, $A \in k$, $B \in k$, and $\overline{AB} \not\subseteq k$

10. D belongs to that side of $\overleftrightarrow{AB}$ which contains C, and to that side of $\overleftrightarrow{BC}$ which contains A, and $\overrightarrow{BD}$ does not cross $\overline{AC}$

SUMMARY

In the preceding pages you discovered many properties of points and lines in a plane. Some of these properties are stated below. We shall, later, refer to these statements as Introduction Axioms. [It is important that you understand what these axioms say, but you should not try to memorize them.]

*

A set which contains at most one point -- that is, a set which either is empty or consists of a single point -- is called a degenerate set. A set which contains more than one point is a nondegenerate set.

One of the first things we noticed about lines is that each line is a nondegenerate set of points. This property of lines and points is expressed by the following axiom:

> Axiom 1.
> Each line contains at least two points.

Given two points, we observed that there is exactly one line which contains them. That is, two points determine a line. Thus, for example, if a line contains three points A, B, and C, then the line determined by A and B is the same as the line determined by B and C. Is this the line determined by A and C? By C and A?

The property that, for each two points there is one and only one line which contains them is expressed by:

> Axiom 2.
> Two points determine a line.

Suppose A and B are two points. Then, by Axiom 2, there is exactly one line which contains A and B. Does the plane contain other points besides those contained in the line determined by A and B? The answer to this question is given by the next axiom.

> <u>Axiom 3.</u>
>
> There are at least three noncollinear points.

Suppose that ℓ and m are two lines, and that each of ℓ and m is parallel to a third line n. Our next axiom tells us that ℓ and m are parallel to each other.

> <u>Axiom 4.</u>
>
> Two lines each parallel to a third line are parallel to each other.

In the preceding pages you studied terms like 'interval', 'segment', 'ray' and 'half-line'. You learned, for example, that the interval joining two points is the set of points between them, and that the union of the interval and the set of its end points is the segment joining these two points. The next axiom tells what these geometric figures are.

> <u>Axiom 5.</u>
>
> $\forall_X \forall_Y$ [that is, for each point X, for each point Y]
>
> $$\begin{aligned} \overline{XY} &= \{Z\colon\ Z \text{ is between } X \text{ and } Y\}, \\ \overset{\bullet\!-\!\bullet}{XY} &= \overline{XY} \cup \{X, Y\}, \\ \overset{\bullet\!\rightarrow}{XY} &= \overset{\bullet\!-\!\bullet}{XY} \cup \{Z\colon\ Y \in \overline{XZ}\}, \\ \overrightarrow{XY} &= \{Z\colon\ Z \in \overset{\bullet\!\rightarrow}{XY} \text{ and } Z \neq X\}, \\ \overleftrightarrow{XY} &= \overrightarrow{YX} \cup \overrightarrow{XY} \end{aligned}$$

Axiom 5 tells you that you can translate, say, '$C \in \overline{AB}$' into 'C is between A and B'. Then, a segment is described in terms of an interval; a ray is described in terms of a segment and the points "beyond" one of its end points; and a half-line is described as a ray that lost its vertex. Finally, for a point A and a point B, the set $\overleftrightarrow{AB}$ is described as the union of two half-lines.

What we need now is an axiom which tells us that if A and B are

two points, the set $\overleftrightarrow{AB}$ is actually the line which Axiom 2 tells us is determined by A and B. The next axiom does this for us and, among other things, implies that for a single point A, $\overleftrightarrow{AA} = \emptyset$. This axiom, Axiom 6, amounts to saying two things:

(1) if $A \neq B$ and A, B, and C are collinear then $C \in \overleftrightarrow{AB}$

and:

(2) if $C \in \overleftrightarrow{AB}$ then $A \neq B$ and A, B, and C are collinear

> <u>Axiom 6.</u>
>
> $\forall_X \forall_Y \forall_Z$
>
> $Z \in \overleftrightarrow{XY}$
>
> if and only if
>
> $X \neq Y$ and X, Y, and Z are collinear

[Do you see that if you leave out the words 'and only if', Axiom 6 tells you (1)? If you leave out the words 'if and', Axiom 6 tells you (2).]

Our next four axioms tell more about betweenness. For example, they tell us that if A is between B and C then $A \neq B$ and $A \neq C$. Also, they tell us that between any two points there is a third. They tell us that the set of points between A and B is precisely the set of points between B and A. Finally, if B is between A and C then C is not between A and B. Notice that the axioms express these ideas in terms of intervals.

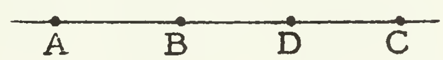
> <u>Axiom 7.</u>
>
> $\forall_X \forall_Y \quad X \notin \overline{XY}$
>
> <u>Axiom 8.</u>
>
> $\forall_X \forall_Y$ if $X \neq Y$ then $\overline{XY} \neq \emptyset$
>
> <u>Axiom 9.</u>
>
> $\forall_X \forall_Y \quad \overline{XY} = \overline{YX}$
>
> <u>Axiom 10.</u>
>
> $\forall_X \forall_Y \forall_Z$ if $Y \in \overline{XZ}$ then $Z \notin \overline{XY}$

[Do you see that Axioms 7 and 9 tell you that no interval contains its end points?]

Suppose A and B are two points. Then, according to Axiom 8, there are points of $\overleftrightarrow{AB}$ between A and B. According to our next axiom, there are also points of $\overleftrightarrow{AB}$ "beyond" B.

> <u>Axiom 11.</u>
>
> $\forall_X \forall_Y$ if $X \neq Y$ then $\{Z\colon\ Y \in \overline{XZ}\} \neq \emptyset$

The next three axioms give you information about the ordering of <u>four</u> points on a line. Suppose B is between A and C. Then, A, B, and C are collinear. Suppose, further, that B is between A and D. Then A, B, and D are collinear. Now, if $C \neq D$, this situation can be shown by one of the following diagrams:

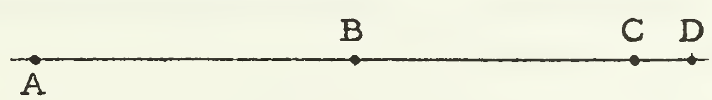


Axiom 12 tells you what we have just observed:

> <u>Axiom 12.</u>
>
> $\forall_W \forall_X \forall_Y \forall_Z$ if $X \in \overline{WY} \cap \overline{WZ}$ and $Y \neq Z$ then $[Y \in \overline{XZ}$ or $Z \in \overline{XY}]$

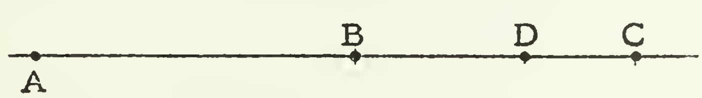
Suppose that B is between A and C:

(a) Then, if C is between B and D:



what can you conclude about A, B, and D?

(b) If D is between B and C:



what can you now conclude about A, B, and D?

Axioms 13 and 14 answer these questions for us:

> <u>Axiom 13.</u>
>
> $\forall_W \forall_X \forall_Y \forall_Z$ if $X \in \overline{WY}$ and $Y \in \overline{XZ}$ then $X \in \overline{WZ}$
>
> <u>Axiom 14.</u>
>
> $\forall_W \forall_X \forall_Y \forall_Z$ if $X \in \overline{WY}$ and $Z \in \overline{XY}$ then $X \in \overline{WZ}$

The last Introduction Axiom contains information about the separation of the plane by a line into two half-planes. One of the things it tells us is that to get from one side of a line to the other, you must cross the line.

> <u>Axiom 15.</u>
>
> The complement of each line is the union of two sets [half-planes] such that
>
> (1) both end points of each segment which does not intersect the line belong to one of the sets, and
>
> (2) if both end points of a segment belong to one of the sets then each point of the segment belongs to this set.

* * *

Compare Axioms 1, 2, and 3 with Messages 3, 4, and 5 on pages 6-1 and 6-2. Jo of Zabranchburg High School was able to deduce Message A [page 6-3] from the earlier messages. Do you think she could have deduced from Axioms 1, 2, and 3 a statement to the effect that there are at least three lines? You try to do it.

SOME CONSEQUENCES OF THE AXIOMS

Just as, in earlier units, you derived theorems about real numbers from the basic principles of Unit 2, so one can derive theorems about geometric figures from the foregoing Introduction Axioms. We shall not prove such theorems in this unit. But, partly to help you to clarify your ideas of the concepts discussed so far, here are some theorems one <u>could</u> derive from Axioms 1-15. Later, we shall refer to theorems like these as <u>Introduction</u> <u>Theorems</u>. [Actually, each of these theorems is quite easily proved. After learning more about proofs, you may find it interesting to see for yourself how some of these theorems follow from the Introduction Axioms.]

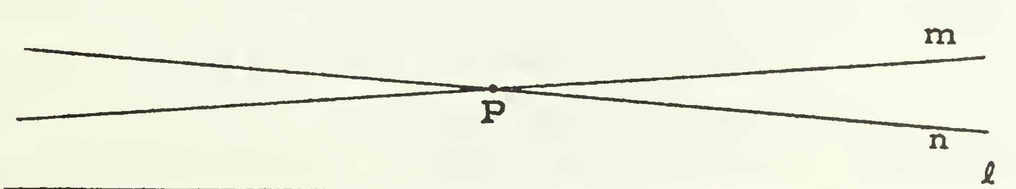
One of the consequences of Axiom 2 is that the intersection of two lines consists of at most one point. For example, if ℓ and m are two lines then either $\ell \cap m = \emptyset$ or $\ell \cap m$ consists of just one point. [If it is not the case that ℓ and m are two lines, that is, if $\ell = m$, what do you know about $\ell \cap m$?] So, the intersection of two lines is a degenerate set. This is expressed by the following theorem:

<u>Theorem 1.</u> The intersection of two lines consists of at most one point.

A consequence of Axiom 3 is:

<u>Theorem 2.</u> For each line, there is a point not on it.

Axiom 4 tell us that if each of two lines is parallel to a third line then the two lines are parallel to each other. Now, suppose ℓ is a line and P is a point not on ℓ. Can there exist <u>two</u> lines through P each of



which is parallel to ℓ? Axiom 4 says that if both m and n are parallel to ℓ then m is parallel to n. But, since $P \in m \cap n$, it follows that $m \cap n \neq \emptyset$. So, m is not parallel to n. Therefore, it is not the case that both m and n are parallel to ℓ. So, we have the following theorem:

> Theorem 3.
>
> For each line ℓ, and for each point $P \notin \ell$, there is at most one line which contains P and is parallel to ℓ.

Axiom 2 together with Axiom 6 tells us that if $A \neq B$ then $\overleftrightarrow{AB}$ is the line determined by the two points A and B. That is,

> Theorem 4.
>
> $\forall_X \forall_Y$ if $X \neq Y$ then $\overleftrightarrow{XY}$ is the line determined by X and Y

Axiom 5 tells us what we mean by terms like 'interval', 'segment', 'half-line', etc. From Axiom 5 we can conclude, for example, that $\overline{AB} \cup \{A, B\}$ is $\overset{\bullet\!-\!\bullet}{AB}$, that $\overset{\bullet\!-\!\bullet}{AB} \subseteq \overset{\bullet\!\rightarrow}{AB}$, and that $\overset{\bullet\!\rightarrow}{AB}$ is $\overrightarrow{AB} \cup \{A\}$. With the help of Axiom 7, we can also conclude that $\overline{AB} \subseteq \overrightarrow{AB}$. Also, $\overrightarrow{AB} \subseteq \overleftrightarrow{AB}$, and $\overleftrightarrow{AB} = \overleftrightarrow{BA}$.

Another consequence of Axiom 5 is that if A and B are two points then $A \in \overrightarrow{BA}$ and $\overrightarrow{AB} \subseteq \overleftrightarrow{AB}$.

Axiom 6 gives us a basis for concluding that $\overleftrightarrow{AB} = \emptyset$ if $A = B$. And, in view of some of the consequences of Axioms 5 and 7 mentioned earlier, we can also say that if $A = B$ then $\overrightarrow{AB}$ and $\overline{AB}$ are empty and $\overset{\bullet\!\rightarrow}{AB} = \{A\}$ and $\overset{\bullet\!-\!\bullet}{AB} = \{A\}$. [So, if A is any point, $\overleftrightarrow{AA}$, $\overrightarrow{AA}$, $\overline{AA}$, $\overset{\bullet\!\rightarrow}{AA}$, and $\overset{\bullet\!-\!\bullet}{AA}$ are degenerate sets.]

Axiom 8 tells us that between two points there must be at least one point. So, if $A \neq B$ then $\overline{AB}$ contains at least one point. Suppose this point is C. Then, with the help of Axioms 9, 13, and 14, we can prove that $\overline{CB} \subseteq \overline{AB}$. We can then use Axioms 7-9 to show that $\overline{AB}$ contains two points. We summarize these results in the next theorem.

Theorem 5.

(a) $\forall_X \forall_Y$ $\overline{XY} \cup \{X, Y\} = \overset{\vdash\!\!\dashv}{XY} \subseteq \overset{\bullet\!\!\rightarrow}{XY} = \overrightarrow{XY} \cup \{X\}$

(b) $\forall_X \forall_Y$ $\overline{XY} \subseteq \overrightarrow{XY} \subseteq \overleftrightarrow{XY} = \overleftrightarrow{YX}$

(c) $\forall_X \forall_Y$ if $X \neq Y$ then $[X \in \overrightarrow{YX}$ and $\overset{\bullet\!\!\rightarrow}{XY} \subseteq \overleftrightarrow{XY}]$

(d) $\forall_X \forall_Y$ if $X = Y$ then $\overleftrightarrow{XY} = \emptyset$ [so, also, $\overrightarrow{XY} = \emptyset = \overline{XY}$ and $\overset{\bullet\!\!\rightarrow}{XY} = \{X\} = \overset{\vdash\!\!\dashv}{XY}$]

(e) $\forall_X \forall_Y \forall_Z$ if $Z \in \overline{XY}$ then $\overline{ZY} \subseteq \overline{XY}$

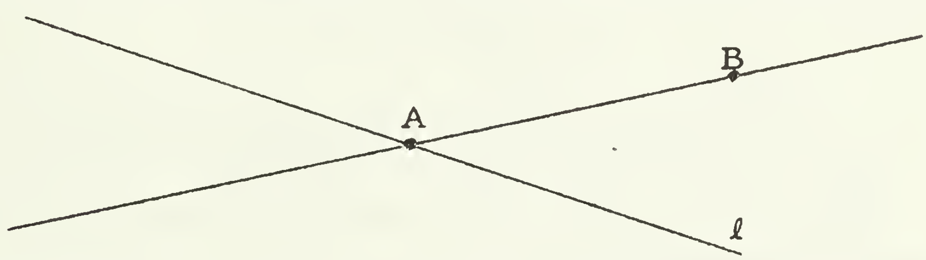
(f) $\forall_X \forall_Y$ if $X \neq Y$ then $\overline{XY}$ is nondegenerate

The following theorem is a consequence of Theorem 4 and Theorem 5d:

Theorem 6.

$\forall_W \forall_X \forall_Y \forall_Z$ if $W \neq Z$ and $\{W, Z\} \subseteq \overleftrightarrow{XY}$ then $\overleftrightarrow{WZ} = \overleftrightarrow{XY}$

Suppose A is a point on ℓ and B is a point not on ℓ. Since $A \neq B$,



the points A and B determine the line $\overleftrightarrow{AB}$. This follows from Theorem 4. Since $\overleftrightarrow{AB} \neq \ell$, it follows from Theorem 1 that $\overleftrightarrow{AB} \cap \ell$ contains at most one point. But, $A \in \ell$ and $A \in \overleftrightarrow{AB}$. So, $\overleftrightarrow{AB} \cap \ell = \{A\}$.

What about $\overrightarrow{AB} \cap \ell$? Since, by Axiom 5, $\overrightarrow{AB} \subseteq \overleftrightarrow{AB}$, it follows that $\overrightarrow{AB} \cap \ell \subseteq \overleftrightarrow{AB} \cap \ell$. So, $\overrightarrow{AB} \cap \ell \subseteq \{A\}$. But, again by Axiom 5, $A \notin \overrightarrow{AB}$. Therefore, $\overrightarrow{AB} \cap \ell = \emptyset$.

We summarize these results in the following theorem:

Theorem 7.

$\forall_\ell \forall_X \forall_Y$ if $X \in \ell$ and $Y \notin \ell$ then $\overleftrightarrow{XY} \cap \ell = \{X\}$ and $\overrightarrow{XY} \cap \ell = \emptyset$

Axioms 5 and 7 and Theorem 5d tell us that if A and B are two

points then the half-line $\overrightarrow{AB}$ is made up of the interval $\overline{AB}$, the point B, and the points "beyond" B.

With the help of Axioms 5 and 9 it now follows that the line $\overleftrightarrow{AB}$ is made up of the points "on the left of" A, the point A, the interval $\overline{AB}$, the point B, and the points "on the right of" B.

In more precise language:

> <u>Theorem 8.</u>
>
> $\forall_X \forall_Y$ if $X \neq Y$ then
>
> $$\overrightarrow{XY} = \overline{XY} \cup \{Y\} \cup \{Z: Y \in \overline{XZ}\}$$
>
> and
>
> $$\overleftrightarrow{XY} = \{Z: X \in \overline{YZ}\} \cup \{X\} \cup \overline{XY} \cup \{Y\} \cup \{Z: Y \in \overline{XZ}\}$$

The description of a line in Theorem 8 together with Axiom 6 imply the following theorem:

> <u>Theorem 9.</u>
>
> $\forall_X \forall_Y \forall_Z$ if X, Y, and Z are three collinear points then $[X \in \overline{YZ}$ or $Z \in \overline{XY}$ or $Y \in \overline{XZ}]$

Theorem 5b and Axioms 6, 7, and 9 imply:

> <u>Theorem 10.</u>
>
> $\forall_X \forall_Y \forall_Z$ if $Z \in \overline{XY}$ then X, Y, and Z are three collinear points

Some more Introduction Theorems are listed below. Together with each theorem is a list of the axioms and theorems which could be used to prove it. Be sure to draw pictures to illustrate each theorem and to help you understand what the theorem says.

<u>Theorem 11.</u> $\forall_W \forall_X \forall_Y$ if $X \in \overline{WY}$ then $\overrightarrow{XY} = \{Z: X \in \overline{WZ}\}$

[Axioms 7, 9, Theorem 8, and Axioms 12, 13, and 14]

<u>Theorem 12.</u> $\forall_X \forall_Y \forall_Z$ $Z \in \overrightarrow{XY}$ if and only if $X \neq Y$ and $\overrightarrow{XZ} = \overrightarrow{XY}$

[Theorems 5c and 5d for the if-part; Theorem 5d, Axioms 9 and 11, and Theorem 11 for the only-if-part]

<u>Corollary.</u> $\forall_X \forall_Y \forall_Z$ if $Z \in \overrightarrow{XY}$ then $Y \in \overrightarrow{XZ}$

[This theorem is an almost immediate consequence of Theorem 12. (To prove it, you also need the result stated in Theorem 5c.) A theorem related in this way to a preceding theorem is customarily called a <u>corollary</u> of the latter. So, this is a corollary of Theorem 12. The second part of Theorem 8 could have been listed as a corollary of the first part.]

<u>Theorem 13.</u> $\forall_X \forall_Y \forall_Z$ if $Z \in \overrightarrow{XY}$ then $\overset{\bullet\!-\!\bullet}{YZ} \subseteq \overrightarrow{XY}$

[Axiom 9, and Theorems 5, 8, and 12]

<u>Theorem 14.</u> $\forall_W \forall_X \forall_Y$ if $X \in \overline{WY}$ then $\overleftrightarrow{XY} = \overrightarrow{XW} \cup \{X\} \cup \overrightarrow{XY}$

[Axioms 5, 7, and 9, and Theorems 8 and 11]

<u>Theorem 15.</u> $\forall_\ell \forall_X \forall_Y$ X and Y are on the same side of ℓ if and only if $\overset{\bullet\!-\!\bullet}{XY} \cap \ell = \emptyset$

[Axiom 15]

<u>Theorem 16.</u> $\forall_\ell \forall_X \forall_Y$ X and Y are on opposite sides of ℓ if and only if $Y \notin \ell$ and $\overline{XY} \cap \ell \neq \emptyset$

[Theorem 7, Axiom 15(2), and Theorem 5a for the if-part; Theorem 5a and Axiom 15(1) for the only-if-part]

<u>Theorem 17.</u> $\forall_{\ell}\forall_{X}\forall_{Y}\forall_{Z}$ if $\overleftrightarrow{YZ} \cap \ell = \emptyset$ and $\overline{XY} \cap \ell \neq \emptyset$
then $\overline{XZ} \cap \ell \neq \emptyset$

[Theorems 5a, 15, and 16]

<u>Theorem 18.</u> $\forall_{\ell}\forall_{X}\forall_{Y}$ if $X \in \ell$ and $Y \notin \ell$ then $\overrightarrow{XY}$ is a subset of one side of ℓ

[Theorems 7, 13, and 15]

<u>Theorem 19.</u> $\forall_{\ell}\forall_{m}$ m crosses ℓ if and only if $m \cap \ell$ consists of a single point

[Axioms 1, 7, 9, and 11, and Theorems 4, 5c, 14, and 16 for the if-part; Theorems 1, 4, 5b, 5d, and 16 for the only-if-part]

<u>Theorem 20.</u> $\forall_{\ell}\forall_{m}\forall_{n}$ if ℓ is parallel to m and n crosses ℓ then n crosses m.

[Theorems 1, 3, and 19]

6.01 <u>Measures of segments.</u> --We can assign numbers to certain parts of geometric figures. This is often convenient in studying properties of such figures. These numbers are called <u>measures</u>.

Here is a picture of a line ℓ and three points, A, B, and C on ℓ, with B between A and C. One way of assigning measures to segments

of ℓ is to use a ruler. Place your ruler with the inch-scale along the picture with the mark for A next to the 4-mark on the ruler. What scale mark is next to the mark for B? If the ruler you are using is like the one we used in drawing the picture, the scale-difference for the two marks [that is, the smaller scale number subtracted from the larger] should be 2.

Now, place your ruler so that some other scale mark is next to the picture of A. Look at the scale mark for B. Do you still get the scale difference 2?

Such scale differences give you a way of assigning measures to segments. The measure assigned in this way to $\overline{AB}$ is 2. With this way of assigning measures, what measure should be assigned to $\overline{BA}$? To $\overline{BC}$? To $\overline{AC}$?

If your ruler had been different from ours, the same procedure would have assigned a different measure to $\overline{AB}$. For example, there are rulers marked with a $\frac{1}{2}$-inch-scale. You can convert your ruler into one like that by imagining each of your scale numbers to be doubled. If you used such a ruler, what measures would you assign to $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$?

There are other scales less simply related to the inch-scale. Perhaps your ruler has a centimeter-scale [a cm-scale] on it. [If not, find one that does.] If you use such a scale, what measures will you assign to $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$?

When assigning measures with the inch-scale, you assigned to $\overline{AB}$ the measure 2 and to $\overline{BC}$ the measure 1.5. We can say what we just said in a much shorter way:

$$\text{inch-m}(\overline{AB}) = 2 \qquad \text{and} \qquad \text{inch-m}(\overline{BC}) = 1.5$$

[Reach 'inch-m(...)' as 'the inch-measure of ...'.]

Fill in the blanks in the following sentences:

inch-m($\overline{AB}$)	= <u>2</u>	inch-m($\overline{BC}$)	= <u>1.5</u>	inch-m($\overline{AC}$)	= <u>3.5</u>
$\frac{1}{2}$-inch-m($\overline{AB}$)	= ____	$\frac{1}{2}$-inch-m($\overline{BC}$)	= ____	$\frac{1}{2}$-inch-m($\overline{AC}$)	= ____
$\frac{1}{8}$-inch-m($\overline{AB}$)	= ____	$\frac{1}{8}$-inch-m($\overline{BC}$)	= ____	$\frac{1}{8}$-inch-m($\overline{AC}$)	= ____
2-inch-m($\overline{AB}$)	= ____	2-inch-m($\overline{BC}$)	= ____	2-inch-m($\overline{AC}$)	= ____
cm-m($\overline{AB}$)	= <u>5.08</u>	cm-m($\overline{BC}$)	= ____	cm-m($\overline{AC}$)	= ____
og-m($\overline{AB}$)	= <u>28</u>	og-m($\overline{BC}$)	= ____	og-m($\overline{AC}$)	= ____

[You may not have heard of og-measure, but you still should be able to fill in the blanks in the last two sentences.]

In the remainder of our study of geometry, we shall make a great deal of use of measures, but it won't matter what unit is used in assigning them. The reason it won't matter is that all measures have certain basic properties, and these properties are the only ones we will need to use. You may have discovered, for example, that whether 'm' means inch-measure, $\frac{1}{2}$-inch-measure, ..., centimeter-measure, or og-measure,

$$m(\overline{AB}) + m(\overline{BC}) = m(\overline{AC})$$

For short, we can write:

$$AB + BC = AC$$

[Note that the symbol 'AB', for example, is being used as an abbreviation for 'm($\overline{AB}$)'.] This is a consequence of an important property of measures which is expressed by the following generalization:

$$(*) \quad \forall_X \forall_Y \forall_Z \ \text{ if } Y \in \overline{XZ} \text{ then } XY + YZ = XZ$$

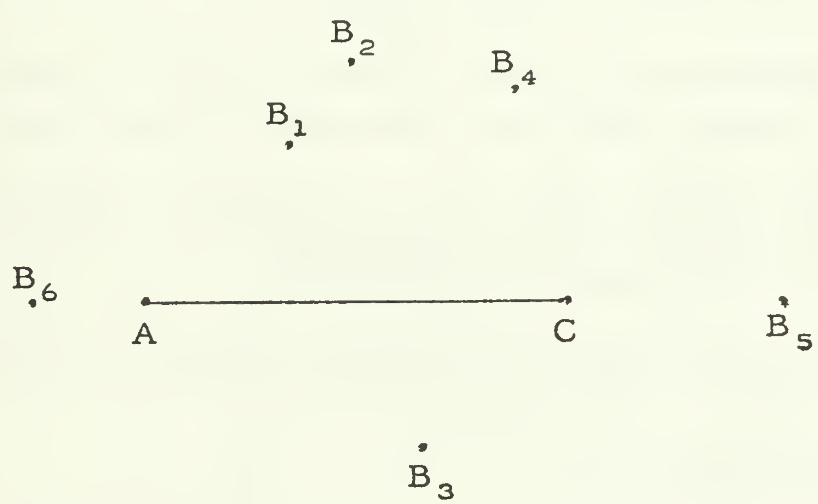
In the example with which we started, A and B were two points and we found various rules for assigning measures to $\overline{AB}$. It is convenient, also, to assign measures to degenerate segments such as $\overline{AA}$. Suppose you use your inch-scale to find the measure of $\overline{AA}$. What do you get? What measure would the other rules assign to $\overline{AA}$? Clearly, since $\overline{AA}$

has only one end point, the same scale mark is opposite each end point, and the measure of $\overline{AA}$ is 0. So, AA + AB = AB. Taking this into account, we can strengthen (*) to:

> <u>Axiom A.</u>
> $\forall_X \forall_Y \forall_Z$ if $Y \in \overline{XZ}$ then $XY + YZ = XZ$

[What is the difference between (*) and Axiom A?]

Let's discover another basic property of measures. Here is a picture of a segment, $\overline{AC}$, and of several other points, B_1, B_2, B_3, ..., none of which belongs to $\overline{AC}$.



Use an inch-scale for assigning measures, and fill out the table below.

i	AB_i	B_iC	$AB_i + B_iC$	AC
1				
2				
3				
4				
5				
6				

What do you notice about the numbers $AB_i + B_iC$ and AC? Could this happen for a point B on $\overline{AC}$?

The work you did on the preceding page suggests the following basic principle:

Axiom B.
$\forall_X \forall_Y \forall_Z$ if $Y \notin \overleftrightarrow{XZ}$ then $XY + YZ > XZ$

EXERCISES

1. Suppose that $P \in \overline{AC}$ and that AP is twice PC. If CA = 18, what is AP?

2. Suppose that $P \in \overline{AB}$ and that $B \in \overline{PC}$. If AC = 24, AP = BC, and PB = 4, what is AB and what is PC?

3. Suppose that A, B, and C are three collinear points such that $AC = 3 \cdot AB$. If BC = 6, what is AB? [There are two solutions!]

4. Suppose that $B \notin \overleftrightarrow{AC}$, AC = 9, and AB = 5. Can BC be 16? Can it be 2? What can you say about BC?

[Supplementary exercises are on page 6-405.]

PROVING THEOREMS

As we have said earlier, one of the things you do in a systematic study of geometry is to derive theorems about geometric figures from the axioms which you adopt as basic principles. So far, we have listed fifteen Introduction Axioms and two axioms concerning measures. We also listed many Introduction Theorems which are derivable from the Introduction Axioms. Now, we shall concentrate on theorems which depend in an essential way on the axioms of measure.

In proving such theorems, we shall be most interested in the way in which the axioms on measure enter into the proofs. Of course, we shall need to use consequences of the Introduction Axioms in our proofs, but we shall not take time to prove them. Also, since measures are numbers, we shall need to use theorems about numbers in our proofs, and, again, we shall not take time to prove these. [Actually, measures are numbers of arithmetic, and the algebra theorems we shall use are

about real numbers. So, as in earlier units, we shall find it convenient to pretend that the measures are nonnegative real numbers.]

Let's consider an example. Here is a very simple theorem:

> <u>Theorem 1-1.</u>
> $\forall_X \; XX = 0$

To see how to begin its proof, we might recall that, so far, Axiom A is the only axiom we have which tells us about equality of measures:

$$\forall_X \forall_Y \forall_Z \text{ if } Y \in \overline{XZ} \text{ then } XY + YZ = XZ$$

Suppose we consider any point A and ask what the measure of $\overline{AA}$ is. Axiom A tells us that if $B \in \overline{AA}$ then $AB + BA = AA$. This might help if we chose a suitable point B in $\overline{AA}$. Now, there isn't much choice, since $\overline{AA} = \{A\}$. So, choosing A, we see that $AA + AA = AA$. A small amount of algebra shows that $AA = 0$. [Can you prove the algebra theorem '$\forall_x$ if $x + x = x$ then $x = 0$'?]

Using this idea, it is not difficult to write a proof of the theorem.

(1)	$\forall_X \forall_Y \forall_Z$ if $Y \in \overline{XZ}$ then $XY + YZ = XZ$	[Axiom A]
(2)	if $A \in \overline{AA}$ then $AA + AA = AA$	[(1)]
(3)	$A \in \overline{AA}$	[Introduction]
(4)	$AA + AA = AA$	[(2) and (3)]
(5)	$AA = 0$	[(4); algebra]
(6)	$\forall_X \; XX = 0$	[(1) - (5)]

Notice that step (2) is an <u>instance</u> of step (1). We indicate this by writing a '(1)' as the marginal comment explaining how we got step (2).

Step (3) is an instance of '$\forall_X \; X \in \overline{XX}$', a theorem which certainly could be derived from the Introduction Axioms. We indicate that this is the case by the marginal comment '[Introduction]'. If we wanted to be more explicit, we would introduce a step between (2) and (3):

(2)	if $A \in \overline{AA}$ then $AA + AA = AA$	
(2.1)	$\forall_X \; X \in \overline{XX}$	[Introduction]
(3)	$A \in \overline{AA}$	[(2.1)]

Now look at step (4). The marginal comment says that it follows from (2) and (3). Do you see that it does?

Notice also, that (5) is a consequence of (4) together with an instance of the algebra theorem '$\forall_x$ if $x + x = x$ then $x = 0$'. We indicate this by the marginal comment '[(4); algebra]'. If we wanted to be more explicit about this, we could have inserted two steps between (4) and (5):

(4)	$AA + AA = AA$	
(4.1)	$\forall_x$ if $x + x = x$ then $x = 0$	[algebra]
(4.2)	if $AA + AA = AA$ then $AA = 0$	[(4.1)]
(5)	$AA = 0$	[(4) and (4.2)]

The marginal comment here tells you that (4.2) is a consequence of (4.1). Explain.

Do you see that the same reasoning that was used to derive (4) from (2) and (3) is used in deriving (5) from (4) and (4.2)?

The final step in the proof is the theorem we wished to prove. The marginal comment for it indicates that steps (1) through (5) form a test-pattern for the generalization.

Also, a glance at the marginal comments shows that (6) is a consequence of Axiom A together with some Introduction and algebra theorems. So, (6) is indeed a theorem.

* * *

> If you have not already studied the Appendix on logic, now is a good time to begin. Pages 6-357 through 6-371 contain a discussion of the rules of reasoning dealing with instances of universal generalizations, test-patterns, substitution, and deriving conclusions from conditional ["if . . . then ____''] sentences.
>
> After completing page 6-371, you may be interested in the discussion of the proof of Theorem 1-1 starting on page 6-398.

* * *

For theorems more complicated [and interesting] than Theorem 1-1, <u>column proofs</u> like that on page 6-33 are likely to be long, and tedious both to write and to recall. So, it is desirable to develop a freer style for writing proofs of a kind which we shall call <u>paragraph proofs</u>.

Here is a paragraph proof for Theorem 1-1:

> Since, for any point A, $A \in \overleftrightarrow{AA}$, it follows from Axiom A that $AA + AA = AA$. So, $AA = 0$. So, $\forall_X\, XX = 0$.

This paragraph proof probably appears much simpler, as it is certainly shorter, than the column proof on page 6-33. This is partly because you have learned something about proving the theorem by studying the column proof. It you knew all about proving theorems, you would be able to write paragraph proofs easily; and we would spend little, if any, time on column proofs. However, you probably have a good many things still to learn about proving theorems; and studying and writing column proofs is one of the best ways to learn these things. Even after you "graduate" to writing paragraph proofs, you will still find it convenient, at times, to expand part of a hard paragraph proof into a column proof, just to make sure that it makes sense. So, for awhile, we shall give you proofs of both kinds to study. Concentrate, first, on learning to write column proofs. But, once you have understood such a proof, try to translate it into a paragraph proof. In this way, you will be learning at the same time to write proofs in both forms, and will most quickly learn how to write acceptable paragraph proofs.

EXERCISES

<u>A.</u> Write a column proof for Theorem 1-1 without looking at the one in the book, and then compare. [If they are different, it doesn't follow that yours is wrong.]

* * *

Theorem 1-1 tells you that the measure of a degenerate segment is 0. That is, if A = B then the measure of $\overleftrightarrow{AB}$ is 0. What if $A \neq B$? What can you say about AB? It certainly seems that $AB > 0$.

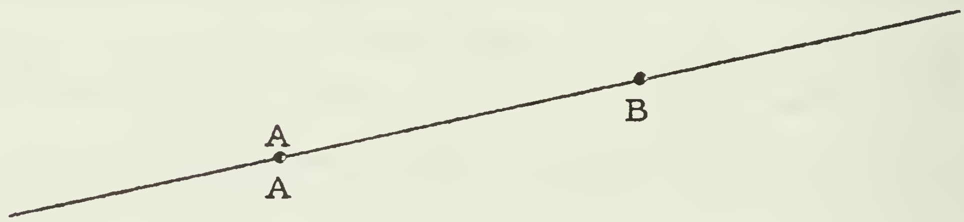
But, does this follow from our axioms? That is, is:

Theorem 1-2.
$\forall_X \forall_Y$ if $X \neq Y$ then $XY > 0$

really a theorem?

Axiom B is the only one of our axioms which deals with inequality of measures; and, in view of Theorem 1-1, we might use an instance of Axiom B like:

$$\text{if } B \notin \overline{AA} \text{ then } AB + BA > AA$$



* * *

B. 1. Here is an incomplete column proof of Theorem 1-2. Complete it, and read the discussion which follows on the next page.

(1)	$A \neq B$	[assumption]*
(2)	______________________	[Axiom B]
(3)	if $B \notin \overline{AA}$ then $AB + BA > AA$	[(2)]
(4)	$B \notin \overline{AA}$	[(1); Introduction]
(5)	$AB + BA > AA$	[(3) and (4)]
(6)	______________________	[Theorem 1-1]
(7)	______________________	[(6)]
(8)	$AB + BA > 0$	[(5) and (7)]
(9)	$AB + AB > 0$	[(8); Introduction]
(10)	$AB > 0$	[(9); algebra]
(11)	if $A \neq B$ then $AB > 0$	[(10); *(1)]
(12)	______________________	[(1) - (11)]

(a) Proving Theorem 1-2 requires that you derive an if-then sentence [that is, a <u>conditional</u> sentence]. Note how this is done in the proof above. The conditional sentence to be derived is:

$$\text{if } A \neq B \text{ then } AB > 0$$

So, you start the proof by assuming the if-part of the conditional. This is step (1). Then, you try to derive from this and other premisses [axioms or previously proved theorems] the then-part of the conditional. This is step (10) in the proof. Step (10) is a consequence of the assumption and other premisses [Axiom B, Introduction and algebra theorems, and Theorem 1-1]. This amounts to saying that the conditional -- step (11) -- is a consequence of just the other premisses. We indicate this by the marginal comment for step (11). We say there that step (11) follows directly from step (10), <u>and</u> since the if-part of step (10) is the assumption (1), we can get rid of or <u>discharge</u> the assumption (1). The two asterisks show this bit of reasoning.

(b) Do you agree that (4) follows from (1) and an instance of a theorem derivable from the Introduction Axioms? State the instance.

(c) The marginal comment for step (8) indicates that it is a consequence of steps (5) and (7). The logic behind this involves substituting one side of the equation (7) for an occurrence the other side of (7) in (5).

(d) What algebra theorem is used in deriving (10) from (9)?

2. Here is the start of a paragraph proof of Theorem 1-2. Finish it, and then compare it with the paragraph proof on page 6-41.

   Suppose that $A \neq B$. Then, $B \notin \overleftrightarrow{AA}$ and, by Axiom B, ...

* * *

For a more detailed discussion on deriving conditional sentences, continue your study of the Appendix through page 6-376. A discussion of the proof of Theorem 1-2 is on page 6-399. For Exercise 4 of that discussion and in preparation for the proofs of Theorem 1-4, you will need to have studied through page 6-386.

* * *

Suppose $B \in \overline{AC}$. It seems pretty clear that AB is less than AC.

Can you prove this? That is, can you prove the following theorem?

> Theorem 1-3.
> $\forall_X \forall_Y \forall_Z$ if $Y \in \overline{XZ}$ then $XY < XZ$

* * *

C. 1. Here is an incomplete column proof of Theorem 1-3. Complete it.

(1)	$B \in \overline{AC}$	[assumption]*
(2)	______________	[Axiom A]
(3)	______________	[(2)]
(4)	$B \in \overleftrightarrow{AC}$	[(1); Introduction]
(5)	______________	[(3) and (4)]
(6)	$B \neq C$	[(1); Introduction]
(7)	______________	[Theorem 1-2]
(8)	______________	[(7)]
(9)	$BC > 0$	[(6) and (8)]
(10)	$AB < AC$	[(5) and (9); algebra]
(11)	______________	[(10); *(1)]
(12)	______________	[(1) - (11)]

2. What instance of an Introduction Theorem could be used (a) in deriving (4) from (1); (b) in deriving (6) from (1)?

☆3. If you have been studying the Appendix, you will want to diagram the proof of Theorem 1-3.

4. Let's look at the algebra used in inferring (10) from (5) and (9).

(5) $AB + BC = AC$

(9) $BC > 0$

From (5) we get:

(9.1) $BC = AC - AB$ [(a) Explain.]

From (9) and (9.1) we get:

(9.2) $AC - AB > 0$ [(b) Explain.]

From (9.2) we get:

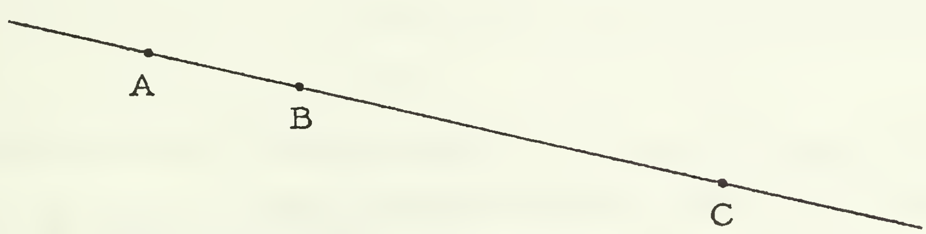
(9.3) $AC > AB$ [(c) Explain.]

which is another way of writing (10).

5. Write a paragraph proof of Theorem 1-3. Compare yours with the one given on page 6-41.

* * *

Axiom A tells us that if $B \in \overline{AC}$ then $AB + BC = AC$.



What about the converse of this? If we are given that $AB + BC = AC$, can we conclude that $B \in \overline{AC}$? The answer is 'yes', and in Part D you will prove:

> <u>Theorem 1-4.</u>
>
> $\forall_X \forall_Y \forall_Z$ if $XY + YZ = XZ$ then $Y \in \overline{XZ}$

* * *

<u>D.</u> 1. Complete the following column proof of Theorem 1-4.

(1) $AB + BC = AC$ [__________]

(2) $AB + BC \not> AC$ [(1); algebra]

(3) ______________________________ [Axiom B]

(4) ______________________________ [(3)]

(5) $B \in \overline{AC}$ [(2) and (4)]

(6) if $AB + BC = AC$ then $B \in \overline{AC}$ [___ ; * _____]

(7) ______________________________ [(1) - (6)]

2. Here is another incomplete column proof of Theorem 1-4. Complete it.

(1) $B \notin \overline{AC}$ [assumption]

(2) ______________________________ [Axiom B]

(3) ______________________________ [(2)]

(4) ______________________________ [(1) and (3)]

(5) $AB + BC \neq AC$ [___ ; algebra]

(6) if $B \notin \overline{AC}$ then _______________ [___ ; _____]

(7) if $AB + BC = AC$ then $B \in \overline{AC}$ [(6)]

(8) ______________________________ [(1) - (7)]

*

<u>Note.</u> The proof in Exercise 1, beginning with the assumption 'AB + BC = AC' is called <u>a</u> <u>direct</u> <u>proof</u> of:

$$\forall_X \forall_Y \forall_Z \text{ if } \underline{XY + YZ = XZ} \text{ then } Y \in \overline{XZ}$$

The proof in Exercise 2, since it begins with the assumption '$B \notin AC$', is called <u>an</u> <u>indirect</u> <u>proof</u> of:

$$\forall_X \forall_Y \forall_Z \text{ if } XY + YZ = XZ \text{ then } \underline{Y \in \overline{XZ}}$$

*

3. (a) Write a paragraph proof for Theorem 1-4 following the reasoning used in Exercise 1. (b) Repeat for Exercise 2. (c) Compare your proofs with those given on page 6-41.

4. Combine Axiom A and Theorem 1-4 into a single statement by using what you know about biconditional [if and only if] sentences.

* * *

For more discussion of some of the rules of reasoning used in the two proofs of Theorem 1-4, continue your study of the Appendix through page 6-389. For a discussion of biconditonal sentences, see pages 6-390 through 6-391.

* * *

Here are paragraph proofs of Theorems 1-2, 1-3, and 1-4. Compare the ones you wrote with these.

Proof of Theorem 1-2:

Suppose that $A \neq B$. Then, $B \notin \overleftrightarrow{AA}$ and, by Axiom B, $AB + BA > AA$. So [since $\overleftrightarrow{BA} = \overleftrightarrow{AB}$], it follows from Theorem 1-1 that $AB + AB > 0$. Thus, $AB > 0$. Hence, if $A \neq B$ then $AB > 0$. Consequently, $\forall_X \forall_Y$ if $X \neq Y$ then $XY > 0$.

Proof of Theorem 1-3:

Suppose that $B \in \overline{AC}$. Then, $B \in \overleftrightarrow{AC}$ and, by Axiom A, $AB + BC = AC$. [So, $BC = AC - AB$.] Also, $B \neq C$ and, by Theorem 1-2, $BC > 0$. Thus, $AC - AB > 0$, and $AB < AC$. Hence, if $B \in \overline{AC}$ then $AB < AC$. Consequently, $\forall_X \forall_Y \forall_Z$ if $Y \in \overline{XZ}$ then $XY < XZ$.

Direct proof of Theorem 1-4:

Suppose that $AB + BC = AC$. Then, $AB + BC \not> AC$. But, by Axiom B, if $B \notin \overleftrightarrow{AC}$ then $AB + BC > AC$. So, $B \in \overleftrightarrow{AC}$. Hence, if $AB + BC = AC$ then $B \in \overleftrightarrow{AC}$. Consequently, $\forall_X \forall_Y \forall_Z$ if $XY + YZ = XZ$ then $Y \in \overleftrightarrow{XZ}$.

Indirect proof of Theorem 1-4:

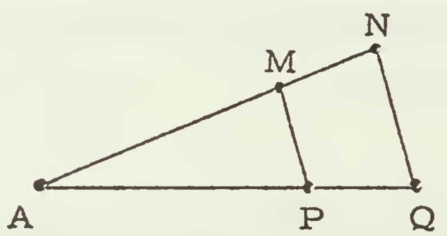
Suppose that $B \notin \overleftrightarrow{AC}$. Then, by Axiom B, $AB + BC > AC$. So, $AB + BC \neq AC$. Hence, if $B \notin \overleftrightarrow{AC}$ then $AB + BC \neq AC$. It follows that if $AB + BC = AC$ then $B \in \overleftrightarrow{AC}$. Consequently, $\forall_X \forall_Y \forall_Z$ if $XY + YZ = XZ$ then $Y \in \overleftrightarrow{XZ}$.

DERIVING A CONCLUSION FROM AN HYPOTHESIS

In each of the exercises which follow, you have a figure, some stated information [called the hypothesis], and a conclusion. Your job is to show that the conclusion follows from

(1) premisses stated in the hypothesis,
(2) additional premisses suggested by the figure,
(3) axioms,
(4) previously proved theorems,
and (5) Introduction and algebra theorems.

Example.



Hypothesis: AM = AP
MN = PQ

Conclusion: AN = AQ

Solution.

(1)	$M \in \overleftrightarrow{AN}$	[figure]
(2)	$\forall_X \forall_Y \forall_Z$ if $Y \in \overline{XZ}$ then $XY + YZ = XZ$	[Axiom A]
(3)	if $M \in \overline{AN}$ then $AM + MN = AN$	[(2)]
(4)	$AM + MN = AN$	[(1) and (3)]
(5)	$AP + PQ = AQ$	[Steps like (1) and (3)]
(6)	$AM = AP$	[Hypothesis]
(7)	$MN = PQ$	[Hypothesis]
(8)	$AM + MN = AP + PQ$	[(6) and (7); algebra]
(9)	$AN = AQ$	[(4), (5), and (8)]

Notice the marginal comment for (1). What it tells you is that (1) is a premiss suggested by the figure. This premiss could have been included in the hypothesis. In fact, if we did include in the hypothesis all such premisses needed in the proof, the figure would really be unnecessary. The reason for giving figures in problems like this is to avoid having to state in the hypothesis premisses concerning such things as collinearity of points and order of points on a line.

Notice, also, the comment for step (5). Can you state the missing steps referred to in the comment? Just as (4) follows from (1) and (3), so (5) follows from a premiss like (1) and an instance of (2) like (3). Here is a reorganization of steps (1) through (9) into a paragraph:

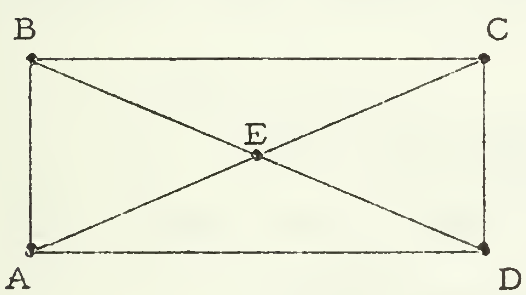
> Since, from the figure, $M \in \overleftrightarrow{AN}$ and $P \in \overleftrightarrow{AQ}$, it follows from Axiom A that $AM + MN = AN$, and $AP + PQ = AQ$. But, by hypothesis, $AM = AP$ and $MN = PQ$. So, $AM + MN = AP + PQ$. Hence, $AN = AQ$.

Notice that steps (1) through (9) make up the major part of the proof of the following theorem:

$$\forall_X \forall_Y \forall_Z \forall_U \forall_V \text{ if } Y \in \overline{XZ},\ U \in \overline{XV},\ XY = XU \text{ and } YZ = UV, \text{ then } XZ = XV$$

EXERCISES

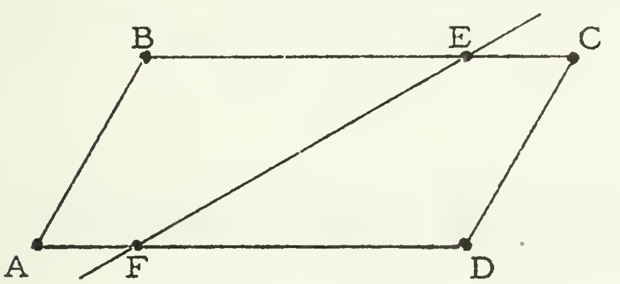
1.



Hypothesis: $AE = EC$, $BE = ED$, $AC = BD$

Conclusion: $AE = ED$

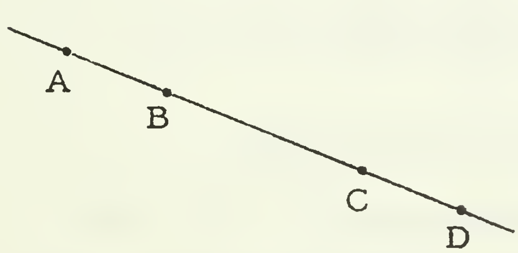
2.



Hypothesis: $AD = BC$, $AF = CE$

Conclusion: $FD = EB$

3.



Hypothesis: $AB = CD$

Conclusion: $AC = BD$

4. Same figure as in Exercise 3, but the Hypothesis and Conclusion are interchanged.

EXPLORATION EXERCISES

A. Draw a line $\overleftrightarrow{CD}$.

1. Mark on it a point E such that $CE < CD$ and $E \in \overline{CD}$.
2. Mark on $\overleftrightarrow{CD}$ a point F such that $CF < CD$ and $F \notin \overline{CD}$.
3. Mark on $\overleftrightarrow{CD}$ a point G such that $CG > CD$ and $D \in \overline{CG}$.
4. Mark on $\overleftrightarrow{CD}$ a point H such that $CH > CD$ and $D \notin \overline{CH}$.
5. Mark on $\overleftrightarrow{CD}$ a point I such that $CI = CD$.

B. Draw a half-line $\overrightarrow{PQ}$.

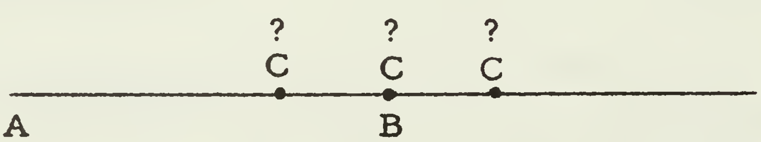
1. Mark on it a point R such that $PR < PQ$.
2. Is it the case that $Q \in \overline{PR}$? That $R = Q$? That $R \in \overline{PQ}$?
3. Mark on $\overrightarrow{PQ}$ a point S such that $S \in \overline{PQ}$.
4. Is it the case that $PS > PQ$? That $PS = PQ$? That $PS < PQ$?

C. Draw a half-line $\overrightarrow{JK}$.

1. Mark on it a point M such that $JM > JK$.
2. Is it the case that $M \in \overline{JK}$? That $M = K$? That $K \in \overline{JM}$?
3. Mark on $\overrightarrow{JK}$ a point N such that $K \in \overline{JN}$.
4. Is it the case that $JN > JK$? That $N = K$? That $JN < JK$?

HALF-LINES

Here is a picture of the half-line $\overrightarrow{AB}$. Suppose that $C \in \overrightarrow{AB}$.



It follows that either $C \in \overline{AB}$ or $C = B$ or $B \in \overline{AC}$.

(1) If $C \in \overline{AB}$ then, by Theorem 1-3, $AC < AB$.
(2) If $C = B$ then, by the principle of identity, $AC = AB$.
(3) If $B \in \overline{AC}$ then, by Theorem 1-3 again, $AC > AB$.

Now, suppose that $AC < AB$. From this it follows that $AC \not> AB$. So, by (3), $B \notin \overline{AC}$. Hence [assuming, as we have done, that $C \in \overrightarrow{AB}$], we conclude that $C \in \overline{AB}$ or $C = B$. But, as also follows from the assumption that $AC < AB$, $AC \neq AB$. So, by (2), $B \neq C$. Hence, from this and our previous conclusion, it follows that $C \in \overline{AB}$. Consequently, if $AC < AB$ then $C \in \overline{AB}$. Combining this with (1) we see that, under our assumption that $C \in \overrightarrow{AB}$, $C \in \overline{AB}$ if and only if $AC < AB$.

So, we have the following theorem:

<u>Theorem 1-5.</u>

$\forall_X\forall_Y\forall_Z$ if $Z \in \overrightarrow{XY}$ then
$Z \in \overline{XY}$ if and only if $XZ < XY$

[For a discussion of the logical rules used in the foregoing proof, continue reading the Appendix through page 6-394. See page 6-400 for a detailed discussion of the proof of Theorem 1-5.]

Similarly we can obtain the following theorem:

<u>Theorem 1-6.</u>

$\forall_X\forall_Y\forall_Z$ if $Z \in \overrightarrow{XY}$ then
$Y \in \overline{XZ}$ if and only if $XZ > XY$

Theorems 1-5 and 1-6 are useful in a situation like the following.

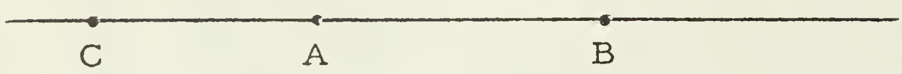
Suppose that $PQ = 5$. Theorem 1-6 tells you that if there is a point $R \in \overrightarrow{PQ}$ such that $PR = 8$ then R is beyond Q. Theorem 1-5 tells you that if there is a point $S \in \overrightarrow{PQ}$ such that $PS = \frac{1}{2} \cdot PQ$ then S is between P and Q. However, neither theorem tells you that there <u>is</u> such a point R or such a point S. For this we need a new axiom:

<u>Axiom C.</u>

$\forall_X\forall_Y\forall_{x > 0}$ if $Y \neq X$
then there is one and only one point
Z such that $Z \in \overrightarrow{XY}$ and $XZ = x$

EXPLORATION EXERCISES

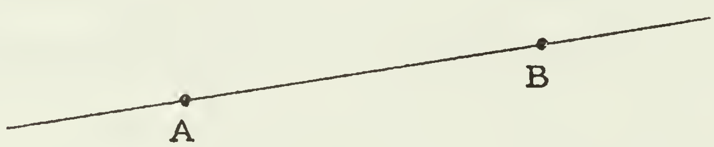
Here is a picture of a line $\overleftrightarrow{CA}$ with a point B on it.



1. Suppose $A \neq B$.
   (a) If $AB = k$, is $k > 0$? Justify your answer.
   (b) Is there a point P such that $P \in \overrightarrow{AB}$ and $AP = \frac{k}{2}$? How many such points are there? What axiom justifies your answer?
   (c) If $P \in \overrightarrow{AB}$, $AB = k$, and $AP = \frac{k}{2}$, where is P with respect to A and B? Justify your answer.
   (d) Mark P on the diagram. Is it the case that $AP = PB$?

2. Suppose $A = B$.
   (a) If $AB = k$, is $k > 0$? Justify your answer.
   (b) Is there a point $P \in \overline{AB}$ such that $AP = \frac{k}{2}$?

MIDPOINTS OF SEGMENTS



Do you think there is a point M which belongs to $\overline{AB}$ and is equidistant from A and B [that is, such that $AM = MB$]? Of course, the answer to this question is 'yes', and, in fact, it follows from our axioms not only that there is such a point but also that there is only one such point. Let's prove this. [By the distance between a point P and a point Q we mean PQ.]

By Axiom A, for each point in $\overline{AB}$, the sum of its distances from A and B is AB. So, a point in $\overline{AB}$ is equidistant from A and B if and only if the distance between it and A is $\frac{1}{2} \cdot AB$. Hence, what we are trying to prove amounts to saying that there is one and only one point $M \in \overline{AB}$ such that $AM = \frac{1}{2} \cdot AB$.

Either $A \neq B$ or $A = B$.

Suppose that $A \neq B$. Then, [by what theorem?] $AB > 0$. So, $0 < \frac{1}{2} \cdot AB < AB$. Since $\frac{1}{2} \cdot AB > 0$, we see from Axiom C that there is one and only one point $M \in \overrightarrow{AB}$ such that $AM = \frac{1}{2} \cdot AB$. Since $M \in \overrightarrow{AB}$ and $AM < AB$, Theorem 1-5 tells us that $M \in \overline{AB}$. So, $M \in \overset{\bullet\!-\!\bullet}{AB}$. Since A is the only point in $\overset{\bullet\!-\!\bullet}{AB}$ which does not belong to $\overrightarrow{AB}$, and since $AA \neq \frac{1}{2} \cdot AB$, M is the only point in $\overset{\bullet\!-\!\bullet}{AB}$ which is equidistant from A and B. So, if $A \neq B$ then there is one and only one point of $\overset{\bullet\!-\!\bullet}{AB}$ which is equidistant from A and B.

On the other hand, if $A = B$ then the only point in $\overset{\bullet\!-\!\bullet}{AB}$ is A. So, [by what theorem?] $AB = 0 = AA$. Thus, $AA = \frac{1}{2} \cdot AB$. So, if $A = B$ then the only point of $\overset{\bullet\!-\!\bullet}{AB}$ which is equidistant from A and B is A.

Consequently, there is one and only one point of $\overset{\bullet\!-\!\bullet}{AB}$ which is equidistant from A and B. The fact that there is just one such point justifies our giving it a name. We shall call it <u>the midpoint of $\overset{\bullet\!-\!\bullet}{AB}$</u>.

We have seen that

(a) the midpoint of a segment belongs to the segment and is equidistant from the end points of the segment, and

(b) any point which belongs to a segment and is equidistant from its end points is the midpoint of the segment.

So, we have the following theorem:

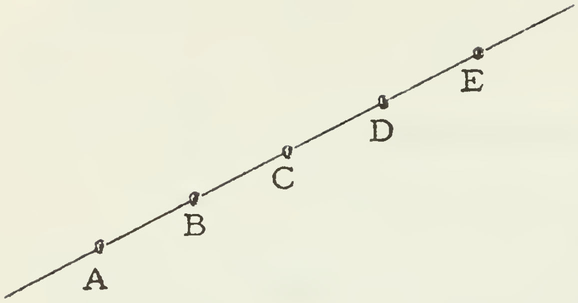
> <u>Theorem 1-7.</u> [Definition of midpoint]
>
> $\forall_X \forall_Y \forall_Z$ Y is the midpoint of $\overset{\bullet\!-\!\bullet}{XZ}$
> if and only if $Y \in \overset{\bullet\!-\!\bullet}{XZ}$ and $XY = YZ$

A theorem which, like this one, characterizes a new term ['midpoint'] is often called <u>a definition</u>. In giving proofs we shall usually not state this theorem. Instead, we shall state the consequences of it which we wish to use. [See the Example on the next page.]

We have also proved:

> <u>Theorem 1-8.</u>
>
> The distance between the midpoint of a segment and either end point of the segment is half the distance between the end points.

Example.



Hypothesis: B is the midpoint of $\overline{AC}$,
C is the midpoint of $\overline{BD}$,
D is the midpoint of $\overline{CE}$,

Conclusion: C is the midpoint of $\overline{AE}$

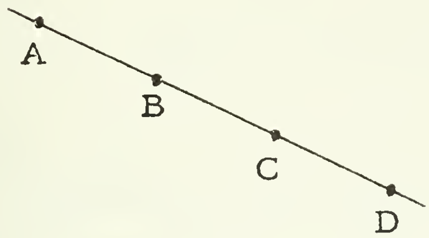
(1)	$B \in \overline{AC}$	
(2)	$C \in \overline{BD}$	[Hypothesis; def. of midpoint]
(3)	$D \in \overline{CE}$	
(4)	$C \in \overline{AE}$	[(1), (2), and (3); Introduction]
(5)	$AB = BC$	
(6)	$BC = CD$	[Hypothesis; def. of midpoint]
(7)	$CD = DE$	
(8)	$\forall_X \forall_Y \forall_Z$ if $Y \in \overline{XZ}$ then $XY + YZ = XZ$	[Axiom A]
(9)	if $B \in \overline{AC}$ then $AB + BC = AC$	[(8)]
(10)	$AB + BC = AC$	[(1) and (9)]
(11)	if $D \in \overline{CE}$ then $CD + DE = CE$	[(8)]
(12)	$CD + DE = CE$	[(3) and (11)]
(13)	$AB = CD$	[(5) and (6)]
(14)	$BC = DE$	[(6) and (7)]
(15)	$CD + DE = AC$	[(10), (13), and (14)]
(16)	$AC = CE$	[(12) and (15)]
(17)	C is the midpoint of $\overline{AE}$	[(4) and (16); def. of midpoint]

Notice how the definition of midpoint is used in (1) and (5). Since 'the midpoint of $\overline{AC}$' is a short way of saying 'the point of $\overline{AC}$ equidistant from A and C', the premiss 'B is the midpoint of $\overline{AC}$' tells you that $B \in \overline{AC}$ and that $AB = BC$.

Now, look at (17). Since 'the midpoint of $\overline{AE}$' is a short way of saying 'the point of $\overline{AE}$ equidistant from A and E', steps (4) and (16) tell you that C is the midpoint of $\overline{AE}$.

EXERCISES

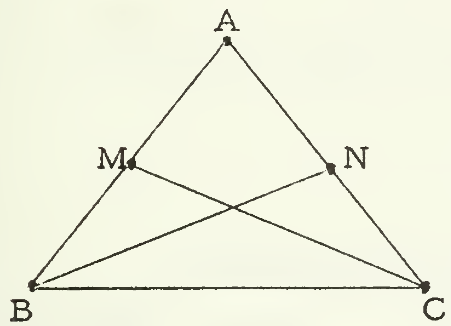
A. 1.



Hypothesis: B is the midpoint of $\overline{AC}$, $AB = CD$

Conclusion: C is the midpoint of $\overline{BD}$

2.



Hypothesis: M is the midpoint of $\overline{AB}$, N is the midpoint of $\overline{AC}$, $AB = AC$

Conclusion: $AM = AN$

B. 1. Suppose that A, M, and B are collinear points, and that $AM = \frac{1}{2} \cdot AB$. Does it follow that M is the midpoint of $\overline{AB}$?

2. Suppose that $AB = 6$, $BC = 4$, M is the midpoint of $\overline{AB}$, and N is the midpoint of $\overline{BC}$.

(a) If B is between A and C then $MN =$ _____ .

(b) If C is between A and B then $MN =$ _____ .

(c) If $C \notin \overleftrightarrow{AB}$ then _____ $< MN <$ _____ .

3. Suppose that C and D are points on $\overrightarrow{AB}$ such that $AC = 5 \cdot AB$ and $AD = 7 \cdot AB$. If M is the midpoint of $\overline{AD}$ then

(a) $AM =$ _____ $\cdot AB$

(b) $BM =$ _____ $\cdot AB$

(c) $MC =$ _____ $\cdot CD$

☆C. Theorem 1-7 tells you that, given a segment $\overline{AB}$, the midpoint of $\overline{AB}$ is the only point which is equidistant from A and B and belongs to $\overline{AB}$. But, suppose that, for some point P, you know only that $AP = PB$ and $P \in \overleftrightarrow{AB}$. Can you conclude that P is the midpoint of $\overline{AB}$? In order to do so, all you need to show is that $P \in \overline{AB}$. By an Introduction Theorem, you know that if $P \in \overleftrightarrow{AB}$ then either

(1) $B \in \overline{AP}$, or (2) $A \in \overline{BP}$, or (3) $P \in \overline{AB}$.

Show that, assuming that AP = PB, case (1) cannot occur. [One way to do this is to show that if $B \in \overline{AP}$ then $AP \neq PB$.] Similarly, show that case (2) cannot occur. Doing so will establish:

> Theorem 1-9.
>
> The only point of a line equidistant from two of its points is the midpoint of the segment with these end points.

SUMMARY OF SECTION 6.01

Notation and terminology

measure of a segment $m(\overline{AB})$ AB midpoint of a segment

Axioms

A. $\forall_X \forall_Y \forall_Z$ if $Y \in \overline{XZ}$ then $XY + YZ = XZ$ [6-31]

B. $\forall_X \forall_Y \forall_Z$ if $Y \notin \overline{XZ}$ then $XY + YZ > XZ$ [6-32]

C. $\forall_X \forall_Y \forall_{x > 0}$ if $Y \neq X$ then there is one and only one point Z such that $Z \in \overrightarrow{XY}$ and $XZ = x$ [6-45]

Theorems

1-1. $\forall_X$ $XX = 0$

1-2. $\forall_X \forall_Y$ if $X \neq Y$ then $XY > 0$

1-3. $\forall_X \forall_Y \forall_Z$ if $Y \in \overline{XZ}$ then $XY < XZ$

1-4. $\forall_X \forall_Y \forall_Z$ if $XY + YZ = XZ$ then $Y \in \overline{XZ}$

1-5. $\forall_X \forall_Y \forall_Z$ if $Z \in \overrightarrow{XY}$ then $Z \in \overline{XY}$ if and only if $XZ < XY$

1-6. $\forall_X \forall_Y \forall_Z$ if $Z \in \overrightarrow{XY}$ then $Y \in \overline{XZ}$ if and only if $XZ > XY$

1-7. $\forall_X \forall_Y \forall_Z$ Y is the midpoint of $\overline{XZ}$ if and only if $Y \in \overline{XZ}$ and $XY = YZ$

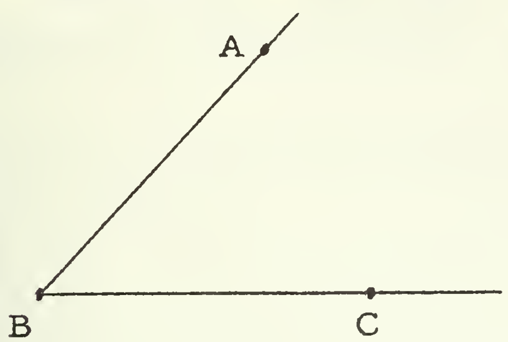
1-8. The distance between the midpoint of a segment and either end point of the segment is half the distance between the end points.

1-9. The only point of a line equidistant from two of its points is the midpoint of the segment with these end points.

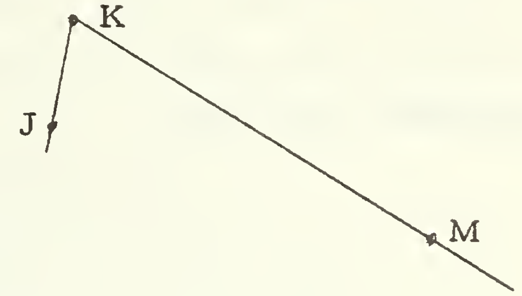
[Supplementary exercises are on page 6-406.]

6.02 <u>Angles and their measures.</u> -- Here are pictures of angles.

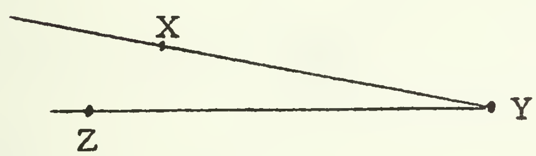
(1)



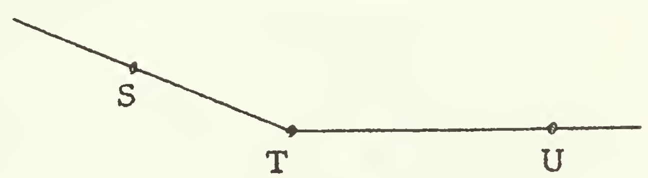
(2)



(3)



(4)



An <u>angle</u> is the union of two noncollinear rays which have the same vertex. The rays are <u>the sides of the angle</u>, and their common vertex is <u>the vertex of the angle</u>.

The angle pictured in (1) is $\overrightarrow{BA} \cup \overrightarrow{BC}$. An abbreviation for '$\overrightarrow{BA} \cup \overrightarrow{BC}$' is '$\angle ABC$'. [Read '$\angle ABC$' as 'angle ABC'.] Since $\overrightarrow{BA} \cup \overrightarrow{BC} = \overrightarrow{BC} \cup \overrightarrow{BA}$, it follows that $\angle ABC = \angle CBA$. The point B is the vertex of $\angle ABC$, and the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are the sides of $\angle ABC$. [Do you see that $\overrightarrow{BA} \cup \overrightarrow{BC} = \overrightarrow{BA} \cup \overrightarrow{BC} = \overrightarrow{BA} \cup \overrightarrow{BC}$?]

Repeat this discussion for the angles in (2), (3), and (4).

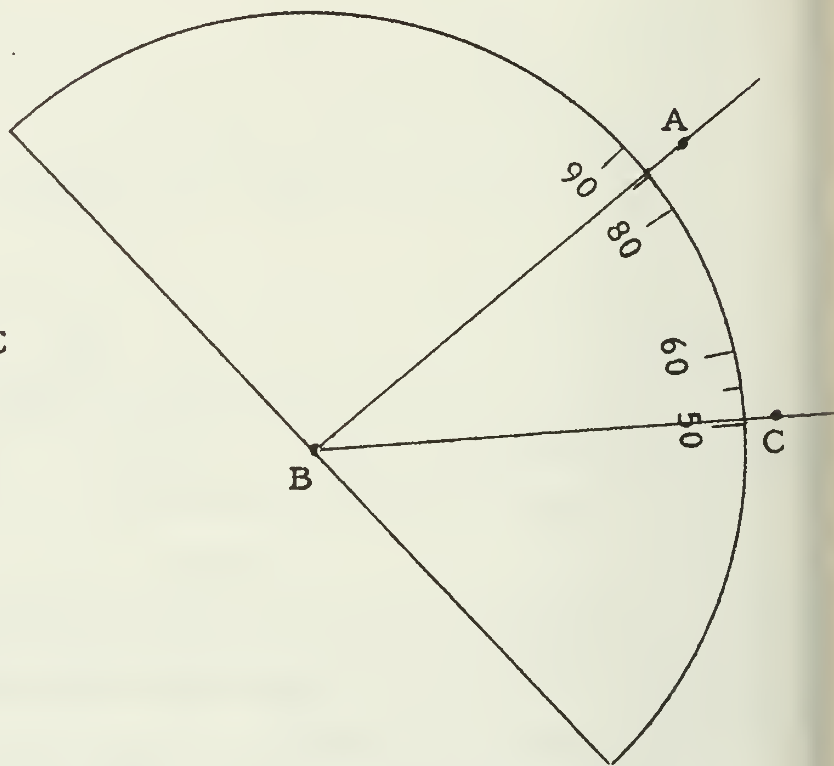
Draw a picture of an angle $\angle KPJ$. What is its vertex? What are its sides? If $V \in \overrightarrow{PK}$ and $W \in \overrightarrow{PJ}$, is $\angle VPW = \angle KPJ$?

Is there an angle, $\angle EFG$, such that $F \in \overline{EG}$? According to our definition, if $F \in \overline{EG}$ [or even if $F \in \overleftrightarrow{EG}$] then $\overrightarrow{FE} \cup \overrightarrow{FG}$ is not an angle. So, the answer is 'no'. Whenever, for example, we use the symbol '$\angle EFG$', you should assume that E, F, and G are noncollinear points.

MEASURE OF AN ANGLE

Just as in the case of segments, we shall find it convenient to assign measures to angles, One way of doing this is to use a <u>protractor</u>. Given an angle, you can assign a measure to the angle by placing your protractor with its center on the picture of the vertex in

such a way that the scale of the protractor is crossed by the pictures [extended, if necessary] of both sides of the angle. Subtract the smaller scale number from the larger. This difference is the <u>degree-measure</u> of the angle.



The degree-measure of $\angle ABC$
$= 86 - 51$
$= 35$

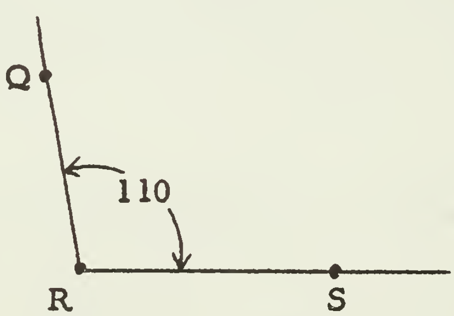
$°m(\angle ABC) = 35$
$°m(\overleftrightarrow{BC} \cup \overleftrightarrow{BA}) = 35$
$\angle ABC$ is an angle of 35°
$\angle B$ is an angle of 35°

As suggested in the diagram, we can abbreviate 'degree-measure of $\angle ABC$' by '$°m(\angle ABC)$'. Since we shall not at present consider other angle-measures, we shall abbreviate this still further to '$m(\angle ABC)$'.

EXERCISES

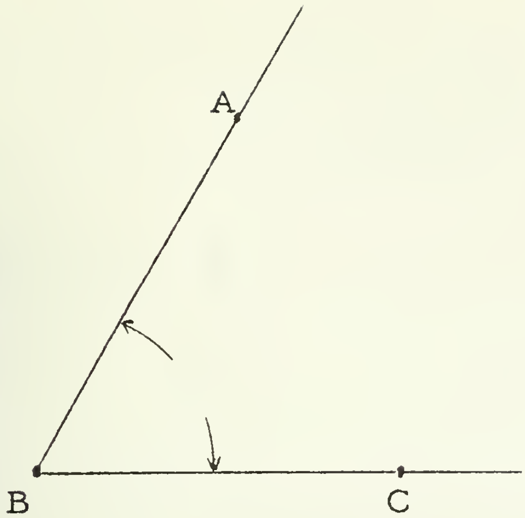
<u>A.</u> Measure each of the angles pictured on the next page, and indicate the measure in at least three ways.

<u>Sample.</u>

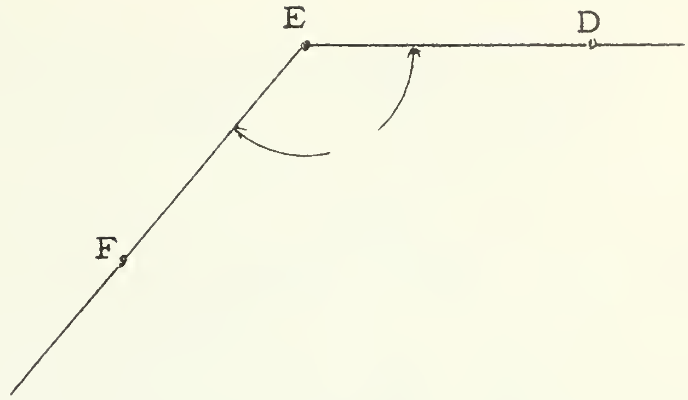


$m(\angle R) = 110$
$m(\angle SRQ) = 110$
$m(\angle QRS) = 110$
$\angle QRS$ is an angle of 110°
$m(\overleftrightarrow{RQ} \cup \overleftrightarrow{RS}) = 110$

1.



2.



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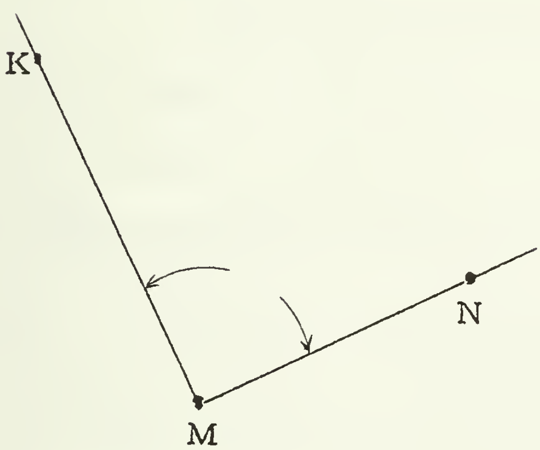
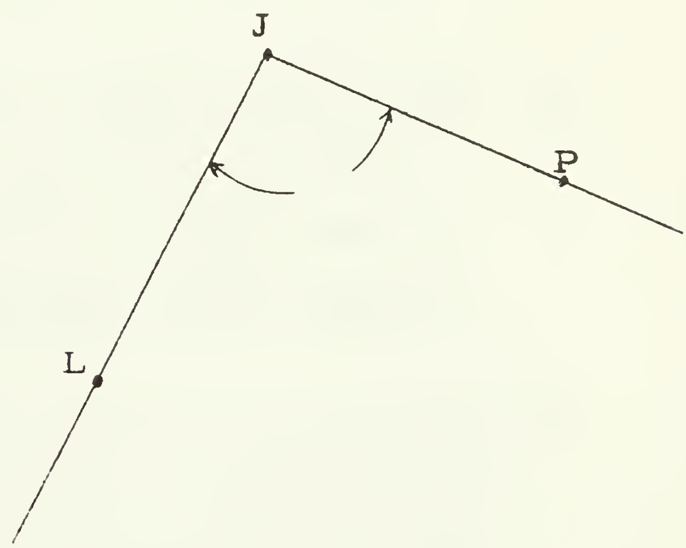


4.



B. Make labeled drawings to illustrate each of the descriptions.

1. $\angle TMP$ is an angle of $30°$

2. $\angle CDF$ is an angle of $45°$ and $\angle DFH$ is an angle of $60°$. [There are actually two situations here, depending on whether C and H are on the same, or different, sides of $\overleftrightarrow{DF}$.]

3. $\angle ABC$ is an angle of $25°$ and $\angle ACB$ is an angle of $40°$ [What is the measure of $\angle BAC$?]

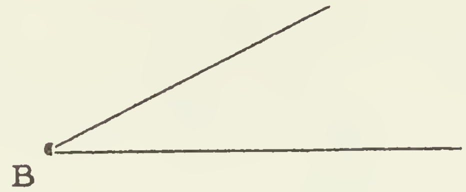
4. $m(\angle QOU) = 35$, $m(\angle UOR) = 20$, and $U \in \overline{QR}$ [What is $m(\angle QOR)$?]

5. $m(\angle QOU) = 35$, $m(\angle UOR) = 40$, and $Q \in \overline{UR}$ [What is $m(\angle QOR)$?]

6. $B \in \overline{AC}$ and h is a half-line contained in one side of $\overleftrightarrow{AC}$ and with vertex B. If $m(h \cup \overrightarrow{BC}) = 19$, what is $m(h \cup \overrightarrow{BA})$?

C. Estimate measures of the given angles without using a protractor.

1.



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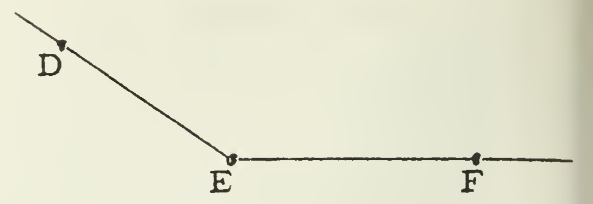

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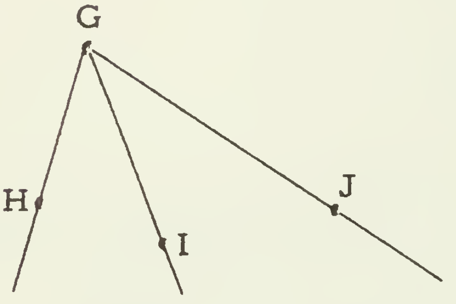

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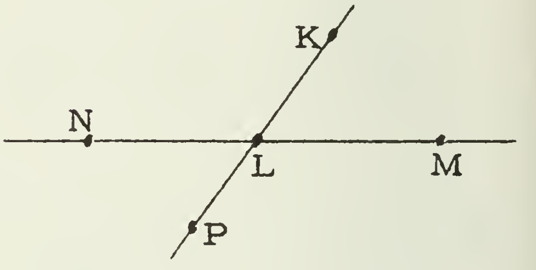


D. Sketch pictures without using ruler and protractor.

1. $\angle ABC$ is an angle of $90°$
2. $\angle T$ is an angle of $45°$
3. $\angle KRS$ is an angle of $135°$
4. $\angle M$ is an angle of $60°$
5. $\angle QHL$ is an angle of $120°$
6. $h \cup \overrightarrow{CD}$ is an angle of $30°$

E. Draw a picture of a line ℓ and mark points A and B on it. Line ℓ separates the plane into two half-planes, s_1 and s_2. Draw a half-line h with vertex A such that $h \subseteq s_1$ and $m(h \cup \overrightarrow{AB}) = 25$. How many such half-lines are there?

* * *

Some of the basic facts you may have discovered about angle-measure are formulated in the next two axioms.

Axiom D.

For each three noncollinear points X, Y, and Z,

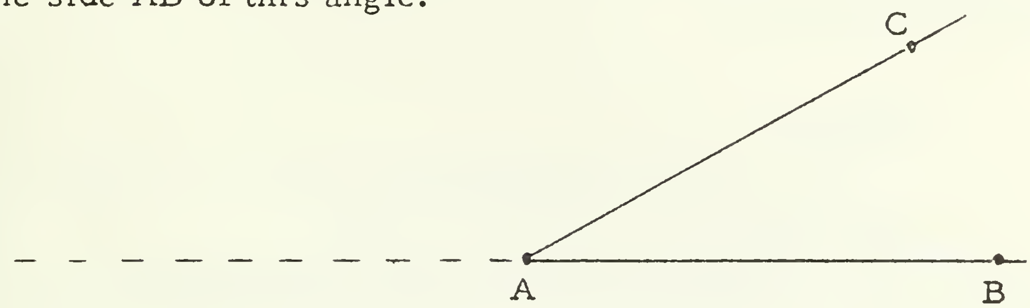
$$0 < {}^{\circ}m(\angle XYZ) < 180.$$

Axiom E.

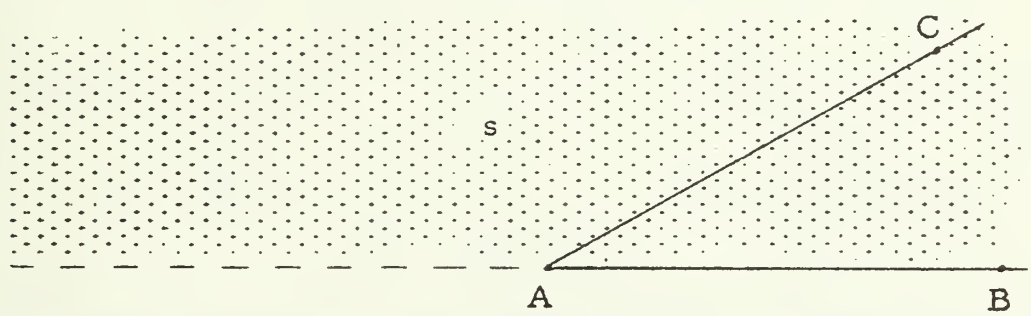
For each two points X and Y, each side s of $\overleftrightarrow{XY}$, and each number x such that $0 < x < 180$, there is one and only one half-line h with vertex X and contained in s such that ${}^{\circ}m(h \cup \overrightarrow{XY}) = x$.

INTERIORS OF ANGLES

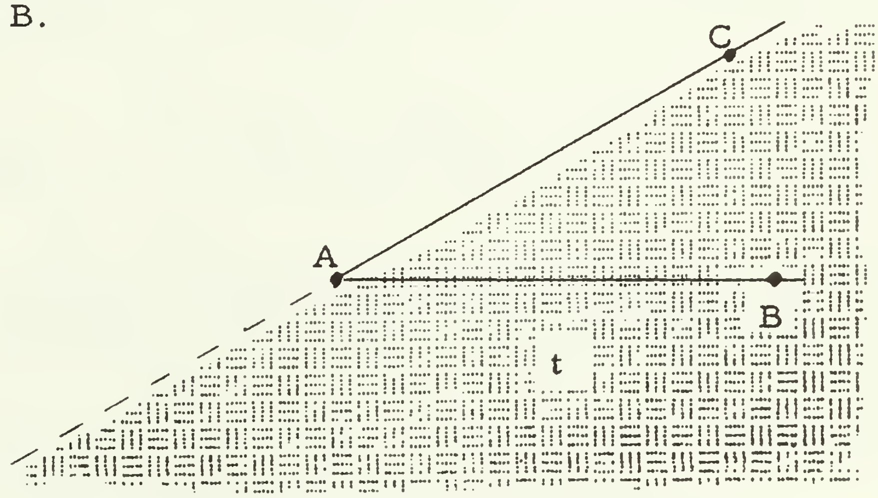
Here is a picture of an angle, $\angle CAB$, and the line $\overleftrightarrow{AB}$ which contains the side $\overrightarrow{AB}$ of this angle.



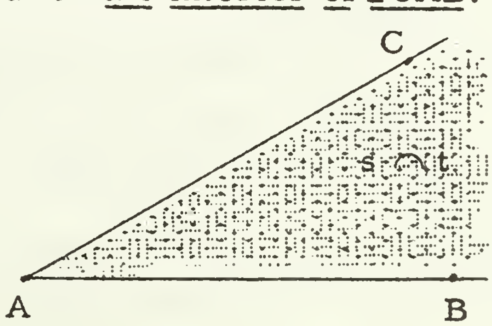
The line $\overleftrightarrow{AB}$ separates the plane into two half-planes one of which, s, contains the point C. [In fact, s contains each point of $\overrightarrow{AC}$.] Also, the



line $\overleftrightarrow{AC}$ separates the plane into two half-planes one of which, t, contains the point B.



The intersection of s and t is called <u>the interior of $\angle CAB$</u>.

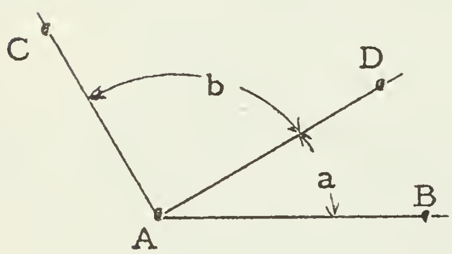


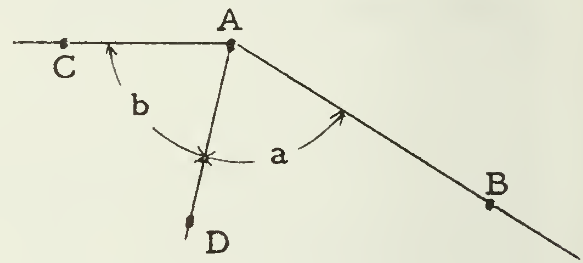
[The <u>exterior</u> of $\angle CAB$ consists of the points which are neither interior to $\angle CAB$ nor on $\angle CAB$.]

Now, suppose D is a point in the interior of $\angle CAB$. If $m(\angle DAB) = a$ and $m(\angle CAD) = b$, what can you say about $m(\angle CAB)$? The next axiom answers this question.

> <u>Axiom F.</u>
>
> For each three noncollinear points X, Y, and Z, and each point W interior to $\angle XYZ$,
>
> $$°m(\angle XYW) + °m(\angle WYZ) = °m(\angle XYZ).$$

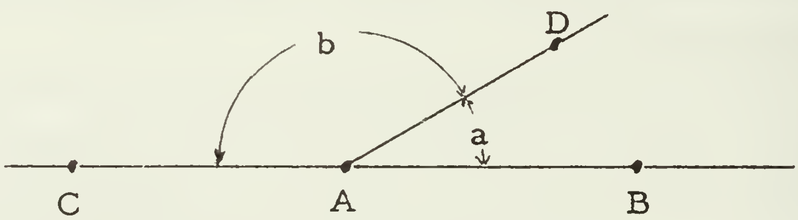
Axiom F takes care of situations like these:





$$m(\angle CAB) = a + b$$

We also need something to take care of a situation like this:



Here, C, A, and B are collinear; so, $\overrightarrow{AC} \cup \overrightarrow{AB}$ is not an angle. Hence, it is nonsense to write '$m(\angle CAB) = a + b$'. However, you have probably discovered when using your protractor that, in a case like this, $a + b = 180$. We formulate this discovery in the following axiom:

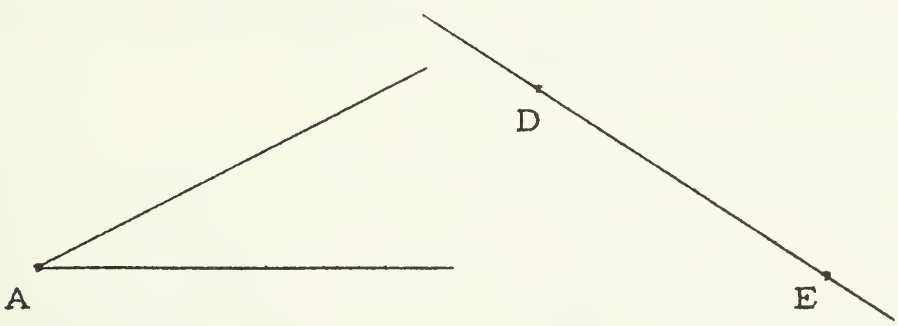
> <u>Axiom G.</u>
>
> For each three noncollinear points X, W, and Z, and for each point $Y \in \overline{XZ}$,
>
> $$°m(\angle XYW) + °m(\angle WYZ) = 180.$$

[Sometimes people say "if $\angle CAB$ is a <u>straight</u> <u>angle</u> then $\angle CAB$ is an angle of 180°". What they mean by this is just what Axiom G says. The reason we do not talk about straight angles is that it is difficult to give a satisfactory definition of angle according to which, when $A \in \overline{BC}$,

$\overrightarrow{AC} \cup \overrightarrow{AB}$ is an angle. If $\overrightarrow{AC} \cup \overrightarrow{AB}$ is an angle, what is its vertex? After all, $\overrightarrow{AC} \cup \overrightarrow{AB}$ is just the line $\overleftrightarrow{CB}$. Given any point on $\overleftrightarrow{CB}$, there are two rays with this point as a vertex whose union is $\overleftrightarrow{CB}$. Each of these points has as good a claim to the title 'vertex of $\overrightarrow{AC} \cup \overrightarrow{AB}$' as any other.]

EXERCISES

A. 1. Here is a picture of $\angle A$, and a picture of $\overleftrightarrow{DE}$. Locate the half-line $\overrightarrow{DF}$ in one of the sides of $\overleftrightarrow{DE}$ such that $m(\angle FDE) = m(\angle A)$.



2. Which axiom tells you that such a half-line exists?

3. Which axiom tells you that [on the side you chose] there is only one such half-line?

B. 1. Draw an angle of 60° and call it '$\angle ABC$'. Locate a point P in the interior of $\angle ABC$ such that $m(\overrightarrow{BP} \cup \overrightarrow{BC}) = 20$. What is $m(\angle PBA)$?

2. Draw an angle of 110° and call it '$\angle MRS$'. Locate a point Q in the exterior of $\angle MRS$ such that $R \in \overline{QS}$. What is $m(\overrightarrow{RQ} \cup \overrightarrow{RM})$?

3. Draw an angle of 30° and call it '$\angle JKL$'. Locate a point T in the exterior of $\angle JKL$ such that $m(\angle TKL) = m(\angle TKJ)$.
   (a) What is $m(\angle TKJ)$?
   (b) If R is a point such that $K \in \overline{TR}$, what is $m(\overrightarrow{KR} \cup \overrightarrow{KJ})$?

4. Suppose $C \notin \overleftrightarrow{AB}$ and $D \in \overline{AB}$. Draw a graph of $\{(x, y)$, x and y are numbers of arithmetic: $x = m(\angle ADC)$ and $y = m(\angle BDC)\}$.

5. (a) Suppose A and B are two points which belong to $\angle MNR$. If $P \in \overline{AB}$, does it follow that P belongs to $\angle MNR$?
   (b) Suppose A and B are two points in the interior of $\angle MNR$.

If $P \in \overline{AB}$, does it follow that P belongs to the interior of $\angle MNR$?

(c) Repeat (b) for the exterior of $\angle MNR$.

6. Suppose x is the measure of $\angle ABC$.

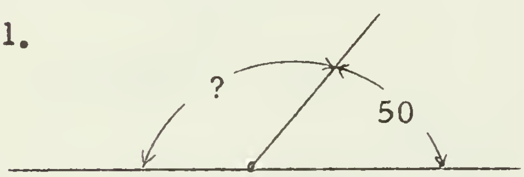
   (a) If P is a point in the interior of $\angle ABC$, what is $m(\angle PBA) + m(\angle PBC)$?

   (b) If Q is a point in the exterior of $\angle ABC$ such that $B \in \overline{QP}$, what is $m(\angle QBA) + m(\angle QBC)$?

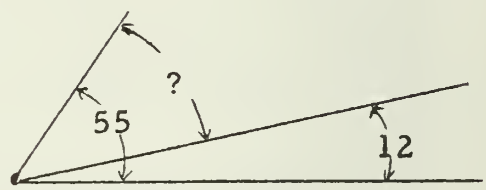
7. Given the ray $\overrightarrow{CD}$, draw the line $\overleftrightarrow{AB}$ such that $C \in \overline{AB}$ and $m(\angle ACD) = 2 \cdot m(\angle BCD)$.

<u>C.</u> Find the indicated measures.

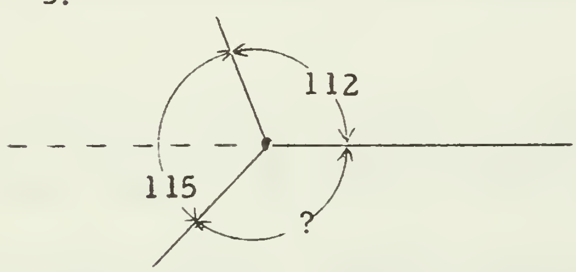
1.



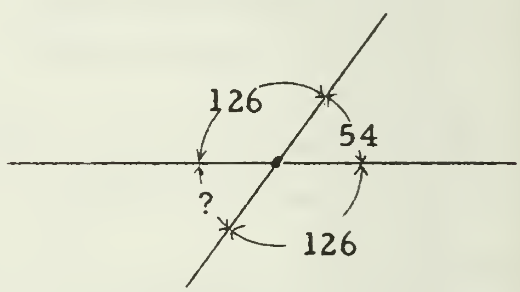
2.



3.



4.



CONGRUENT ANGLES -- CONGRUENT SEGMENTS

Angles are said to be <u>congruent</u> if and only if they have the same measure. Segments, also, are said to be <u>congruent</u> just if they have the same measure. So, if we know, for example, that $m(\angle ABC) = m(\angle D)$, we can say that $\angle ABC$ and $\angle D$ are congruent. More briefly, we can write:

$\angle ABC \cong \angle D$ or: $\angle D \cong \angle ABC$

[The symbol '$\cong$' is read as 'is congruent to' or as 'is congruent with'.]

Similarly, each of the sentences:

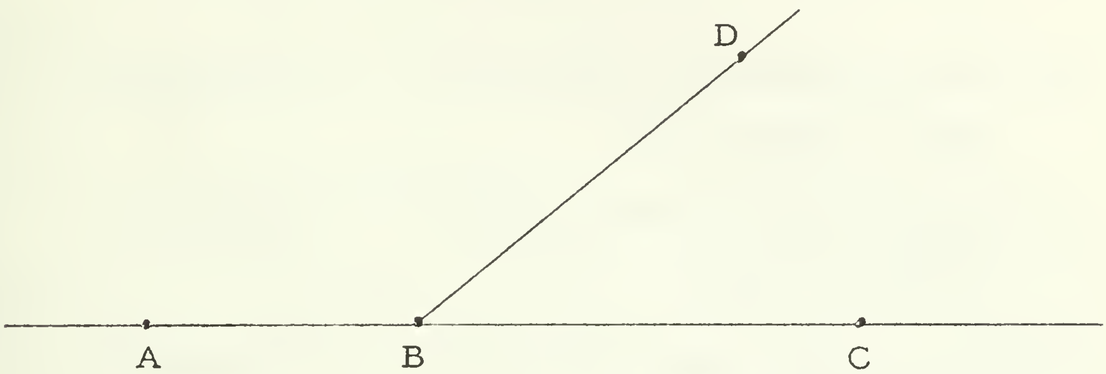
$m(\overline{AB}) = m(\overline{CD})$ and: $AB = CD$

is equivalent to:

$$\overline{AB} \cong \overline{CD}$$

SUPPLEMENTARY ANGLES

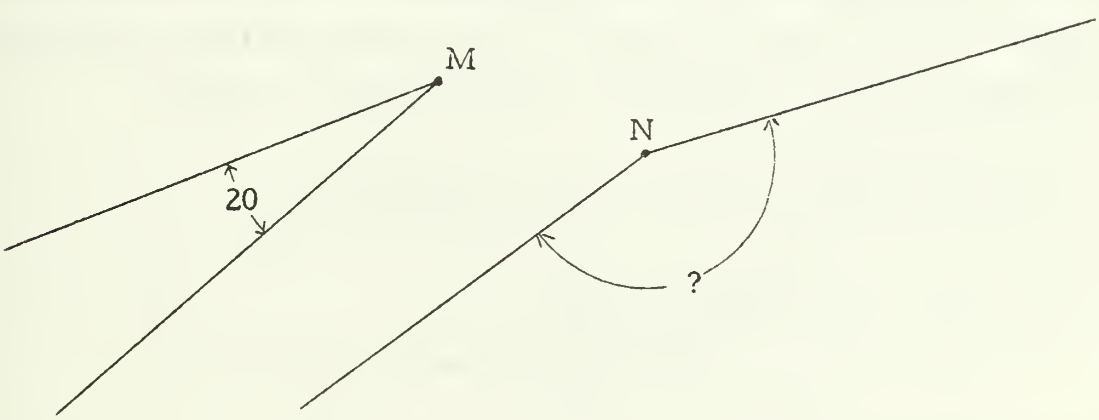
Consider the pair of angles, $\angle CBD$ and $\angle ABD$, where $B \in \overline{AC}$. Axiom G tells us that the sum of the measures of these angles is 180. Because



their measures add up to 180, $\angle CBD$ is said to be a supplement of $\angle ABD$, and $\angle ABD$ is said to be a supplement of $\angle CBD$. In general, $\angle P$ is <u>a supplement of</u> $\angle Q$ if and only if $m(\angle P) + m(\angle Q) = 180$. So, any angle congruent to $\angle CBD$ is a supplement of $\angle ABD$, and any angle congruent to $\angle ABD$ is a supplement of $\angle CBD$.

Two angles each of which is a supplement of the other are called <u>supplementary angles</u>.

Here are pictures of two supplementary angles:



$\angle N$ is one supplement of $\angle M$. How many supplements does $\angle M$ have? What are their measures?

Can an angle be its own supplement? An angle which is its own supplement is called <u>a right angle</u>. Do you see that from this and the definition of supplementary angles it follows that an angle is a right angle if and only if it is an angle of $90°$? The proof of this is easy and illustrates the use of the substitution rule for biconditional sentences. [See page 6-396.] We give the proof on the next page.

(1) $\angle A$ is a right angle if and only if $\angle A$ is a supplement of $\angle A$ [def. of right angle]

(2) $\angle A$ is a supplement of $\angle A$ if and only if $m(\angle A) + m(\angle A) = 180$ [def. of supplementary angles]

(3) $\angle A$ is a right angle if and only if $m(\angle A) + m(\angle A) = 180$ [(1) and (2)]

(4) $m(\angle A) + m(\angle A) = 180$ if and only if $m(\angle A) = 90$ [algebra]

(5) $\angle A$ is a right angle if and only if $m(\angle A) = 90$ [(3) and (4)]

(6) **Theorem 2-1.** An angle is a right angle if and only if it is an angle of $90°$. [(1) - (5)]

We could have defined a right angle to be an angle of $90°$. In that case, we could have proved that an angle is a right angle if and only if it is its own supplement. The test-pattern for this proof would consist of steps (5), (4), (3), (2), and (1) in that order [but the marginal comments would be different]. Write out such a proof.

Here is another theorem about right angles:

Theorem 2-2. All right angles are congruent.

In order to see how to prove the theorem, one must recognise that it stands for a universal generalization of a conditional sentence:

For each $\angle X$, for each $\angle Y$,
if $\angle X$ is a right angle and $\angle Y$ is a right angle then $\angle X \cong \angle Y$.

On the next page is a column proof of Theorem 2-2. Complete the marginal comments.

(1) $\angle A$ is a right angle and $\angle B$ is a right angle	[______________]
(2) An angle is a right angle if and only if it is an angle of 90°.	[Theorem 2-1]
(3) $\angle A$ is a right angle if and only if $m(\angle A) = 90$	[______________]
(4) if $\angle A$ is a right angle then $m(\angle A) = 90$	[______________]
(5) $\angle A$ is a right angle	[(1)]
(6) $m(\angle A) = 90$	[______________]
(7) $m(\angle B) = 90$	[steps like _____]
(8) $m(\angle A) = m(\angle B)$	[(6) and (7)]
(9) $\angle A \cong \angle B$	[(8); def. of congruent angles]
(10) if $\angle A$ is a right angle and $\angle B$ is a right angle then $\angle A \cong \angle B$	[_____ ; _______]
(11) All right angles are congruent.	[______________]

We could simplify this proof by making use of an easily justified rule of reasoning. You know that a universal generalization implies each of its instances. [(2) implies (3).] Also, a biconditional sentence implies each of the conditionals of which it is composed [(3) implies (4)]. So, from a universal generalization of a biconditional sentence, we are justified in inferring either of the conditional sentences which make up any of its instances. So, we could have inferred (4) directly from (2).

*

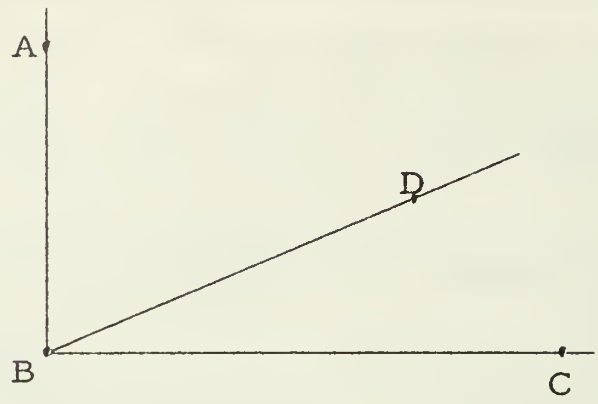
To keep in practice, try writing a paragraph proof of Theorem 2-2 before reading the one we are about to give.

*

> Suppose that $\angle A$ and $\angle B$ are right angles. Then, by Theorem 2-1, $\angle A$ and $\angle B$ are angles of 90°. So, since they have the same measure, they are congruent. Consequently, all right angles are congruent.

COMPLEMENTARY ANGLES

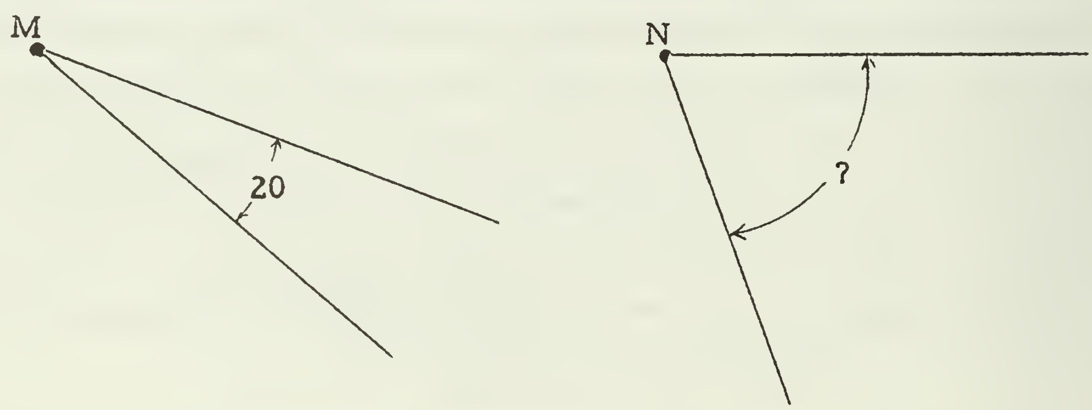
Here is a picture of a right angle $\angle ABC$, and a half-line $\overrightarrow{BD}$ in its interior:



Since $\angle ABC$ is a right angle, $m(\angle ABC) = 90$. So, by Axiom F, we know that the sum of the measures of $\angle CBD$ and $\angle ABD$ is 90. $\angle CBD$ is a complement of $\angle ABD$, and $\angle ABD$ is a complement of $\angle CBD$. In general, $\angle P$ <u>is</u> <u>a</u> <u>complement</u> <u>of</u> $\angle Q$ if and only if $m(\angle P) + m(\angle Q) = 90$. So, any angle congruent to $\angle CBD$ is a complement of $\angle ABD$, and any angle congruent to $\angle ABD$ is a complement of $\angle CBD$.

Two angles each of which is a complement of the other are called <u>complementary</u> <u>angles</u>.

Here are pictures of two complementary angles:



$\angle N$ is a complement of $\angle M$. How many complements does $\angle M$ have? What are their measures?

Does each angle have a supplement? Does each angle have a complement?

$\angle A$ is <u>an</u> <u>acute</u> <u>angle</u> if and only if $m(\angle A) < 90$. $\angle A$ is <u>an</u> <u>obtuse</u> <u>angle</u> if and only if $m(\angle A) > 90$. Do you see that an angle has a complement if and only if it is acute? Explain.

EXERCISES

1. Draw an angle $\angle ABC$. Without measuring $\angle ABC$, draw two of its supplements.

2. Draw a right angle. Now, draw a half-line in this picture so that you then have a picture of two acute angles.

3. Repeat Exercise 2 for obtuse angles.

4. Can two obtuse angles be supplementary? Can two acute angles be supplementary?

5. Suppose $\angle A$ is an angle of $50°$. What is the measure of a complement of $\angle A$? What is the measure of a supplement of $\angle A$?

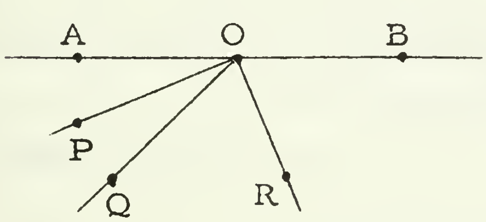
6. Suppose k is the measure of an acute angle, $\angle A$. What is the measure of a supplement of $\angle A$? Of a complement of $\angle A$? What is the difference between the measures of the supplement and the complement of $\angle A$? Illustrate with a figure.

7. Suppose $\angle BAC$ and $\angle CAD$ are complementary angles. Does it follow that $\angle BAD$ is a right angle?

8. If $\angle B$ is its own complement, what is $m(\angle B)$?

9. Draw $\overleftrightarrow{AB}$ and mark a point O such that $O \in \overline{AB}$. Mark a point P in one side of $\overleftrightarrow{AB}$ and a point Q in the other side such that $\angle POQ$ is a right angle. Draw $\angle POQ$. Name a pair of complementary angles in the figure. Name two pairs of supplementary angles.

10. Repeat Exercise 9, but mark P and Q in the same side of $\overleftrightarrow{AB}$.

11. 

Suppose that $O \in \overline{AB}$ and P, Q, and R are on the same side of $\overleftrightarrow{AB}$. Also, suppose that $\angle AOP \cong \angle POQ$ and $\angle QOR \cong \angle ROB$. Find the measure of $\angle POR$.

12. If the measure of an angle is eight times the measure of its supplement, then the angle is an angle of how many degrees?

[Supplementary exercises are on page 6-409.]

* * *

Suppose that $\angle A \cong \angle B$, that $\angle C$ is a supplement of $\angle A$, and that $\angle D$ is a supplement of $\angle B$. What can you say about $\angle C$ and $\angle D$?

Since $\angle A \cong \angle B$, it follows that $m(\angle A) = m(\angle B)$. Since $\angle C$ is a supplement of $\angle A$, $m(\angle C) = 180 - m(\angle A)$. Also, since $\angle D$ is a supplement of $\angle B$, $m(\angle D) = 180 - m(\angle B)$. So, $m(\angle C) = m(\angle D)$. That is, $\angle C \cong \angle D$. So, if $\angle A \cong \angle B$, $\angle C$ is a supplement of $\angle A$, and $\angle D$ is a supplement of $\angle B$, then $\angle C \cong \angle D$. This proves the following theorem:

> <u>Theorem 2-3.</u>
> Supplements of the same angle or of congruent angles are congruent.

The foregoing is a paragraph proof of Theorem 2-3. For practice, let's write a column proof. [We'll write part of it; you write the rest.]

First, let's rewrite Theorem 2-3 as a universal generalization of a conditional sentence:

For all angles $\angle X$, $\angle Y$, $\angle U$, $\angle V$, if $\angle X \cong \angle Y$ and $\angle U$ is a supplement of $\angle X$ and $\angle V$ is a supplement of $\angle Y$, then $\angle U \cong \angle V$.

(1) $\angle A \cong \angle B$ and $\angle C$ is a supplement of $\angle A$ and $\angle D$ is a supplement of $\angle B$ [______________]

(2) $m(\angle A) = m(\angle B)$ [___ ; def. of ___]

(3) $m(\angle A) + m(\angle C) = 180$ [___ ; def. of ___]

(4) $m(\angle B) + m(\angle D) = 180$ [___ ; def. of ___]

(5) $m(\angle C) = m(\angle D)$ [__________; algebra]

(6) if $\angle A \cong \angle B$ and $\angle C$ is a supplement of $\angle A$ and $\angle D$ is a supplement of $\angle B$ then $m(\angle C) = m(\angle D)$ [____ ; * __________]

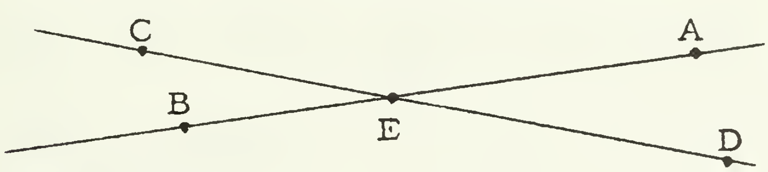
(7) Supplements of the same angle or of congruent angles are congruent. [____ ; def. of _____]

Write a paragraph proof or a column proof for the next theorem:

> Theorem 2-4.
> Complements of the same angle or of congruent angles are congruent.

VERTICAL ANGLES

Here is a picture of two intersecting lines:



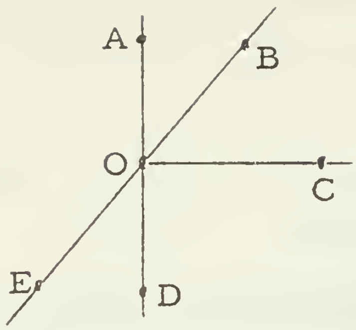
Since $E \in \overline{CD}$ and $A \notin \overleftrightarrow{CD}$, it follows from Axiom G that $\angle AED$ is a supplement of $\angle AEC$. Similarly, $\angle BEC$ is a supplement of $\angle AEC$. It follows from Theorem 2-3 that $\angle AED \cong \angle BEC$. Similarly, it follows that $\angle AEC \cong \angle DEB$.

Whenever we have two intersecting lines, we also have four angles whose common vertex is the point of intersection. Given one of these angles, it shares a side with each of two others, and intersects the fourth in the common vertex. The fourth angle is <u>the vertical angle</u> of the given angle. In general, the vertical angle of $\angle PQR$ is the angle which is a subset of $\overleftrightarrow{PQ} \cup \overleftrightarrow{QR}$ and whose intersection with $\angle PQR$ consists just of the point Q. If one of a pair of angles is the vertical angle of the other then the angles are called <u>vertical angles</u>. So, we have proved in the preceding paragraph that

> Theorem 2-5.
> Vertical angles are congruent.

EXERCISES

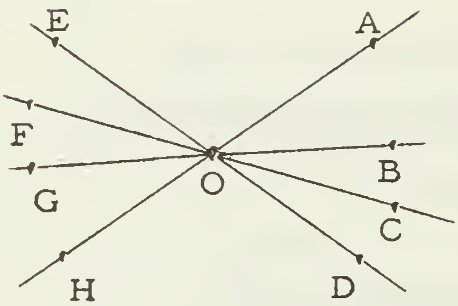
A. 1.



Given: $\angle AOC$ is a right angle,
$O \in \overline{EB}$, $O \in \overline{AD}$,
$m(\angle AOB) = 40$

Find: $m(\angle BOC)$,
$m(\angle EOD)$,
$m(\angle EOC)$

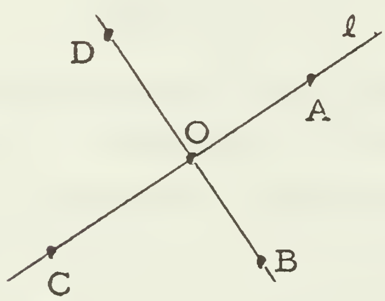
2.



Given: $O \in \overline{AH}$, $O \in \overline{BG}$,
$O \in \overline{GF}$, $O \in \overline{DE}$,
$m(\angle EOF) = 20$,
$\angle AOB \cong \angle BOC$,
$m(\angle AOE) = 110$

Find: the measures of $\angle FOG$, $\angle GOH$, $\angle HOC$, and $\angle HOB$

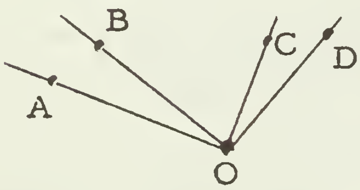
3.



Given: $O \in \overline{AC}$, $O \in \overline{BD}$,
$m(\angle AOB) = 90$

Find: $m(\angle AOD)$,
$m(\angle DOC)$,
$m(\angle COB)$

B. Finish the column proof.



Hypothesis: $\angle BOD$ and $\angle AOC$ are right angles

Conclusion: $\angle AOB \cong \angle COD$

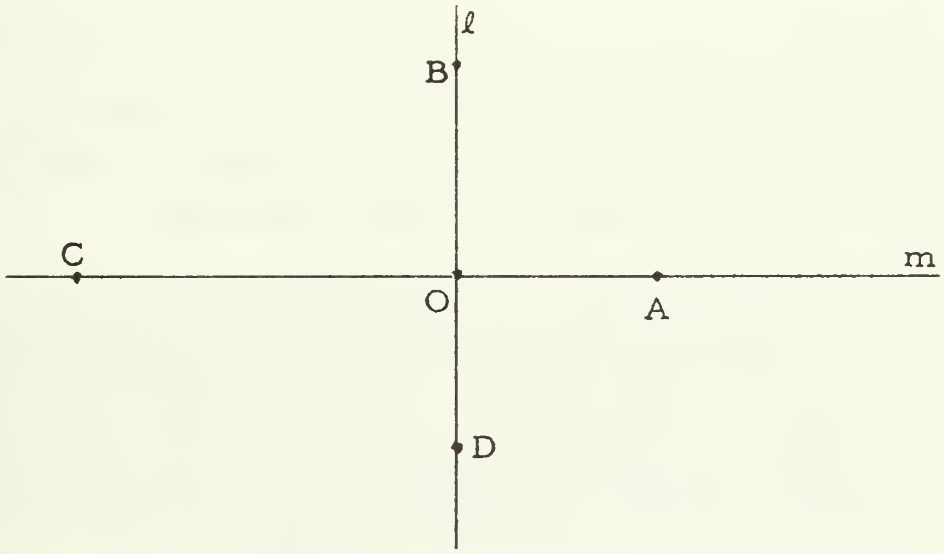
(1)	______________________	[Axiom F]
(2)	B, O, and D are three noncollinear points	[figure]
(3)	______________________	[figure]
(4)	$m(\angle BOC) + m(\angle COD) = m(\angle BOD)$	[(1), (2), and (3)]

C. Prove:

> Theorem 2-6.
> If two supplementary angles are congruent, they are right angles.

PERPENDICULAR LINES

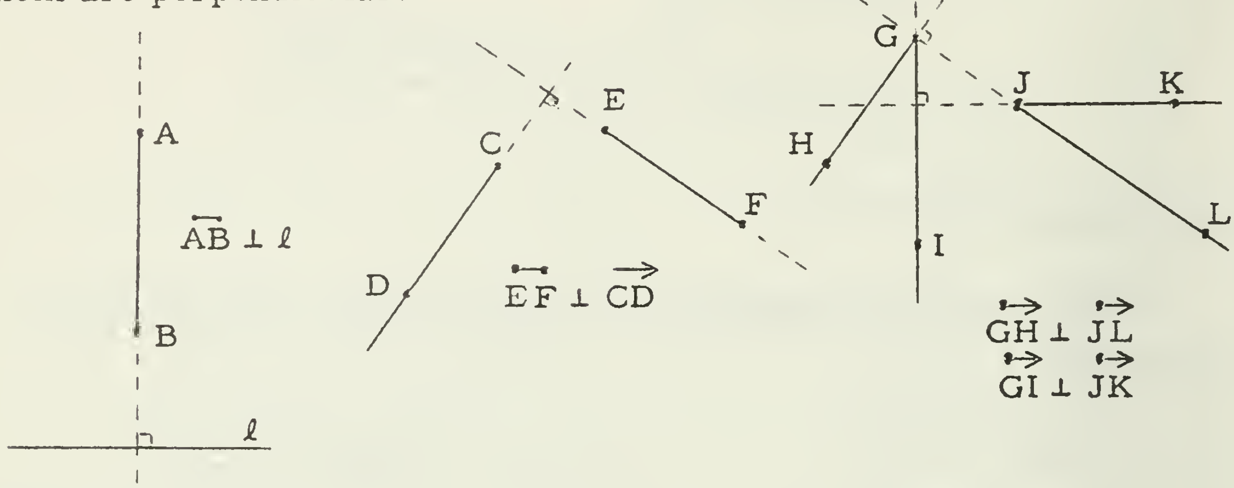
Two lines are perpendicular if and only if their union contains a right angle.



The picture shows two perpendicular lines, ℓ and m. [For short: ℓ ⊥ m. Read '⊥' as 'is perpendicular to'.] Since ℓ and m are perpendicular, one of the angles, ∠AOB, ∠BOC, ∠COD, and ∠DOA, is a right angle. Suppose that ∠AOB is a right angle. Then, since ∠AOB and ∠COD are vertical angles, it follows that ∠COD is a right angle. Now, it follows from Axiom G that m(∠BOC) + m(∠AOB) = 180. Since m(∠AOB) is 90, so is m(∠BOC). Therefore, ∠BOC is a right angle. Finally, ∠BOC and ∠DOA are vertical angles; so, ∠DOA is a right angle. This gives us the following theorem:

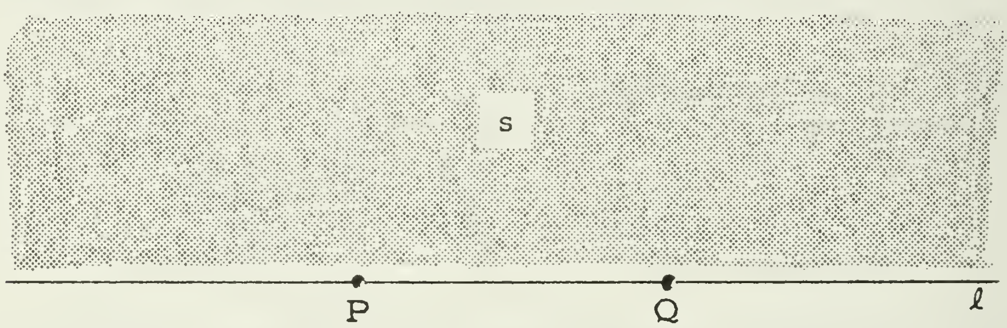
> Theorem 2-7.
> Each of the four angles contained in the union of two perpendicular lines is a right angle.

It will save quite a few words if we agree right now to say that any two sets each of which is either a nondegenerate segment, a half-line, a ray, or a line, are perpendicular if and only if the lines which contain them are perpendicular.



[Notice the use of a "corner" to indicate a right angle.]

Consider a line ℓ and a point P on it. Suppose s is one of the half-planes determined by ℓ, and Q is another point on ℓ.



By Axiom E and Theorem 2-1, there is one and only one half-line h with vertex P and contained in s such that h $\cup$ $\overrightarrow{PQ}$ is a right angle. The line m which contains the half-line h is perpendicular to ℓ. [Why?]

[Use a protractor to help you draw the line m.]

Is there another line which contains P and is perpendicular to ℓ? Such a line would intersect s in a half-line k different from h and such that k $\cup$ $\overrightarrow{PQ}$ is a right angle. But, Axiom E has already told us that h is the only half-line which meets this condition. So, line m is the only line perpendicular to ℓ at P.

Theorem 2-8. At each point on a line, there is one and only one perpendicular.

EXERCISES

1. Draw a line ℓ, and locate points A and B on ℓ such that $\overline{AB}$ is about 1 inch long.

2. Draw the line perpendicular to ℓ at A. Locate a point C on this line such that AC = AB.

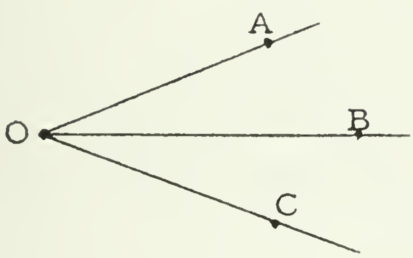
3. Draw the line perpendicular to $\overleftrightarrow{AC}$ at C. Locate a point D on this line such that CD = AC and D is in the interior of $\angle$BAC.

4. Draw the line perpendicular to $\overleftrightarrow{CD}$ at D.

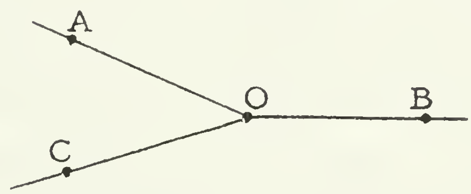
ADJACENT ANGLES

Here are several figures each of which shows a pair of angles, $\angle$AOB and $\angle$BOC:

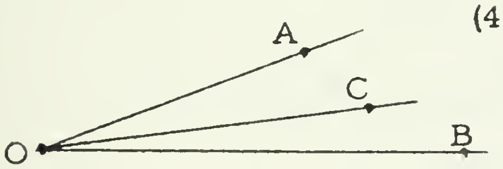
(1)



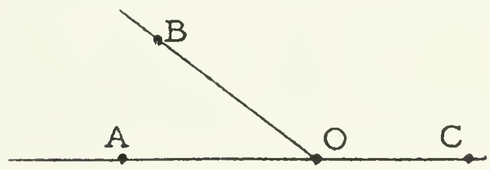
(2)



(3)

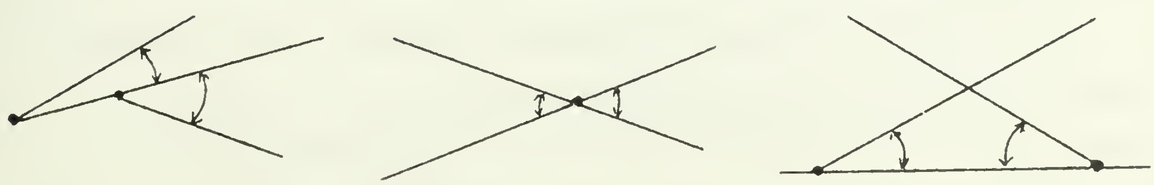


(4)



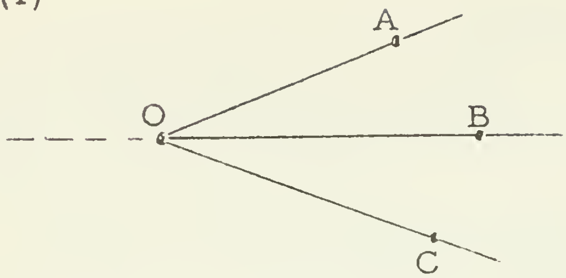
Figures (1), (2), and (4) show cases in which $\angle$AOB and $\angle$BOC are <u>adjacent</u> <u>angles</u>; figure (3) shows a case in which they are not adjacent angles.

Given two angles, a first step in determining if they are adjacent angles is to see if they have a common side. All four cases shown above satisfy this requirement, but here are figures showing pairs of angles which do not have a common side:

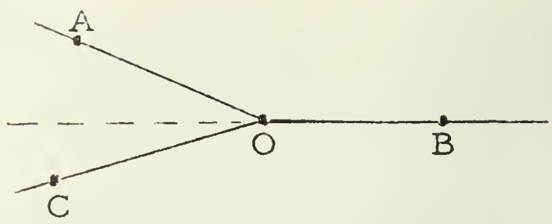


So, none of these pairs of angles is a pair of adjacent angles.

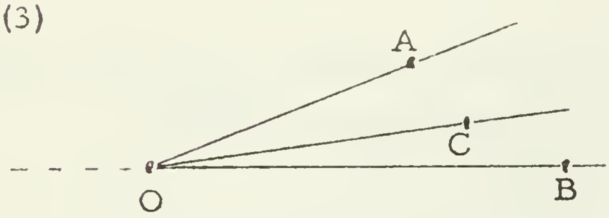
(1)



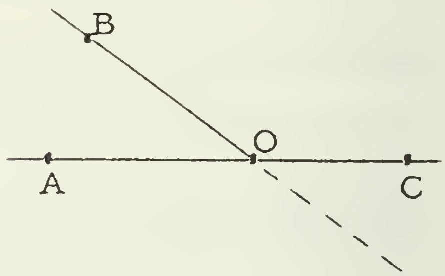
(2)



(3)

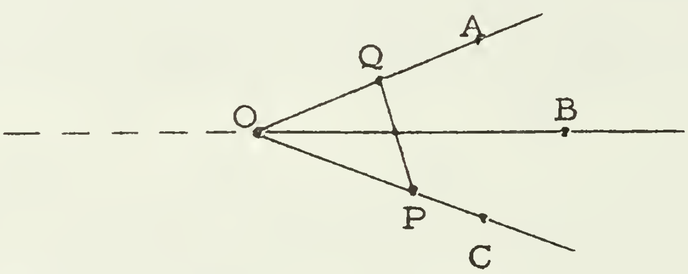


(4)



Two angles are <u>adjacent</u> <u>angles</u> if and only if they have a common side and the noncommon sides are contained in opposite closed half-planes determined by the line containing the common side.

Hence, given two angles which do have a common side, such as $\angle$AOB and $\angle$BOC in the figures above, you can determine whether they are adjacent angles by determining if the noncommon sides are contained in opposite closed half-planes. Let's apply this test to $\angle$AOB and $\angle$BOC shown in figure (1):



Pick two points P and Q, one on each angle and neither on the common side. Since $\overline{PQ} \cap \overleftrightarrow{OB} \neq \emptyset$, P and Q are on opposite sides of $\overleftrightarrow{OB}$. So, $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are contained in opposite closed half-planes. Hence, $\angle$AOB and $\angle$BOC are adjacent angles.

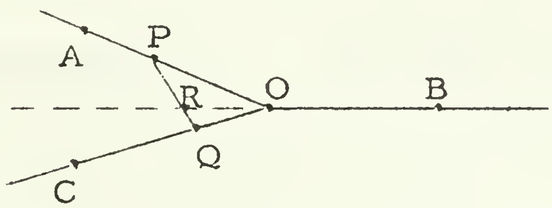
THE SUM OF THE MEASURES OF ADJACENT ANGLES

Pictures (1), (2), and (4) show the possible cases of adjacency of angles. The test-interval will intersect the line containing the common side either in a point of the common side different from the vertex [(1)], or in the vertex [(4)], or in a point not on the common side [(2)]. In each case, we can say something about the sum of the measures of the adjacent angles.

In case (1), Axiom F tells us that the sum of the measures of the adjacent angles is the measure of the angle which is the union of their noncommon sides.

In case (4), Axiom G tells us that the sum of their measures is 180.

Case (2) is a bit more complicated. Can you guess how to find the



sum of the measures of the adjacent angles if you know $m(\angle AOC)$? In this case, if R is the point of intersection of the test-interval and $\overleftrightarrow{OB}$, then, by Axiom F, $m(\angle AOR) + m(\angle ROC) = m(\angle AOC)$. But, by Axiom G, $m(\angle AOR) = 180 - m(\angle AOB)$, and $m(\angle ROC) = 180 - m(\angle BOC)$. So, $360 - [m(\angle AOB) + m(\angle BOC)] = m(\angle AOC)$. Consequently, in case (2), the sum of the measures of the adjacent angles is 360 minus the measure of the angle which is the union of their noncommon sides.

Notice that in case (1) the sum of the measures of the adjacent angles is less than 180, and in case (2) the sum is greater than 180. [Why?] So, if the sum of the measures of adjacent angles is 180, the noncommon sides must be collinear [case (4) is all that is left]. In other words,

> if adjacent angles are supplementary,
> then their noncommon sides are collinear.

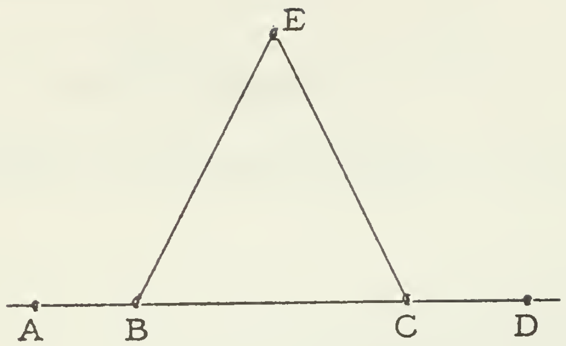
On the other hand, Axiom G and the definition of supplementary angles tell us that

> if the noncommon sides of adjacent angles are collinear,
> then the angles are supplementary.

Combining these two results, we obtain the following theorem:

> Theorem 2-9.
>
> Adjacent angles are supplementary if and only if their noncommon sides are collinear.

Example.



Hypothesis: $\angle EBC \cong \angle ECB$, $B \in \overline{AC}$, $C \in \overline{BD}$

Conclusion: $\angle ABE \cong \angle DCE$

Solution.

(1)	$B \in \overline{AC}$	[Hypothesis]
(2)	$\angle ABE$ and $\angle EBC$ are adjacent angles whose noncommon sides are collinear	[(1); def. of adjacent angles; Introduction]
(3)	Adjacent angles are supplementary if and only if their noncommon sides are collinear.	[Theorem 2-9]
(4)	$\angle ABE$ and $\angle EBC$ are supplementary	[(2) and the if-part of (3)]
(5)	$\angle DCE$ and $\angle ECB$ are supplementary	[Steps like (1) - (4)]
(6)	$\angle EBC \cong \angle ECB$	[Hypothesis]
(7)	Supplements of the same angle or of congruent angles are congruent.	[Theorem 2-3]
(8)	$\angle ABE \cong \angle DCE$	[(4), (5), (6), (7)]

Note that we have shortened the proof by omitting instances of (3) and (7). You may follow this practice in writing your own column proofs.

Compare this proof with the following paragraph proof:

Since, by hypothesis, $B \in \overline{AC}$, it follows that $\angle ABE$ and $\angle EBC$ are adjacent angles whose noncommon sides are collinear. So, by Theorem 2-9, they are supplementary. Similarly, since $C \in \overline{BD}$, it follows that $\angle DCE$ and $\angle ECB$ are supplementary. But, by hypothesis, $\angle EBC \cong \angle ECB$. Hence, by Theorem 2-3, $\angle ABE \cong \angle DCE$.

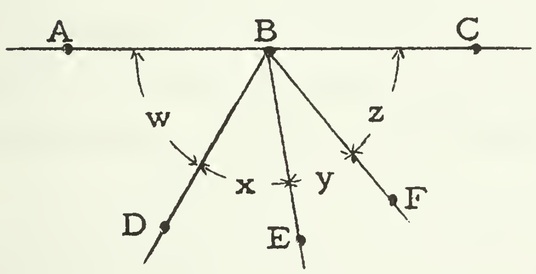
One of the important differences between a column proof and a paragraph proof is that in writing a column proof, you write the theorems [steps (3) and (7)] used as steps in the proof, but in writing a paragraph proof, you usually just refer to the theorems ["by Theorem 2-9", "by Theorem 2-3"] .

Now, if you don't have the textbook at hand when you are writing a paragraph proof [for example, on a test], it may not be possible to refer to the theorems by number. In that case, you can say something like 'by an earlier theorem' or 'by a previously proved theorem', and then state the theorems as footnotes following the proof. Your teacher may want to decide on the style for doing this.

EXERCISES

Write column proofs or paragraph proofs.

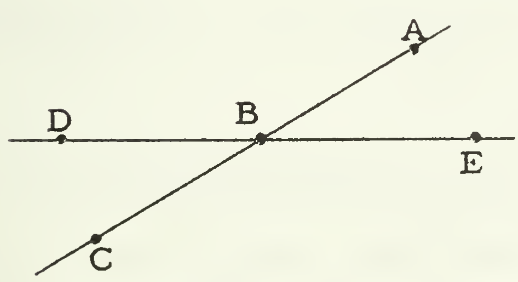
1.



Hypothesis: $w + y = 90$, $x + z = 90$

Conclusion: A, B, and C are collinear

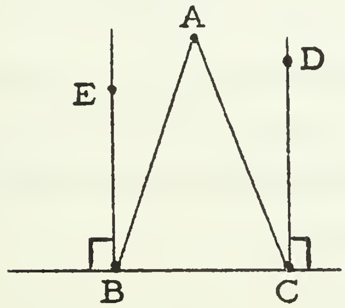
2.



Hypothesis: $\angle ABE \cong \angle DBC$, A, B, and C are collinear

Conclusion: D, B, and E are collinear

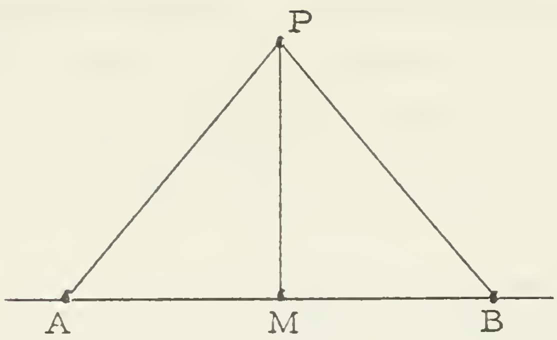
3.



Hypothesis: $\angle ABC \cong \angle ACB$, $\overleftrightarrow{DC} \perp \overleftrightarrow{CB}$, $\overleftrightarrow{EB} \perp \overleftrightarrow{BC}$

Conclusion: $\angle EBA \cong \angle DCA$

4.



Hypothesis: $M \in \overline{AB}$,
$\angle BMP \cong \angle AMP$

Conclusion: $\overleftrightarrow{PM} \perp \overleftrightarrow{AB}$

[Supplementary exercises are on page 6-411.]

EXPLORATION EXERCISES

1. Draw two acute angles with the same measure, one with vertex A and the other with vertex A′. Choose a side of each angle, and on the chosen sides, mark new points B and B′, respectively, such that AB = A′B′. [What axiom tells you that once B is chosen, there is a unique point B′ which satisfies the given conditions?] On the other sides of the angles, mark new points C and C′, respectively, such that AC = A′C′. Measure $\overline{BC}$ and $\overline{B'C'}$. Do they have the same measure?

2. Repeat Exercise 1, but now draw two obtuse angles with the same measure.

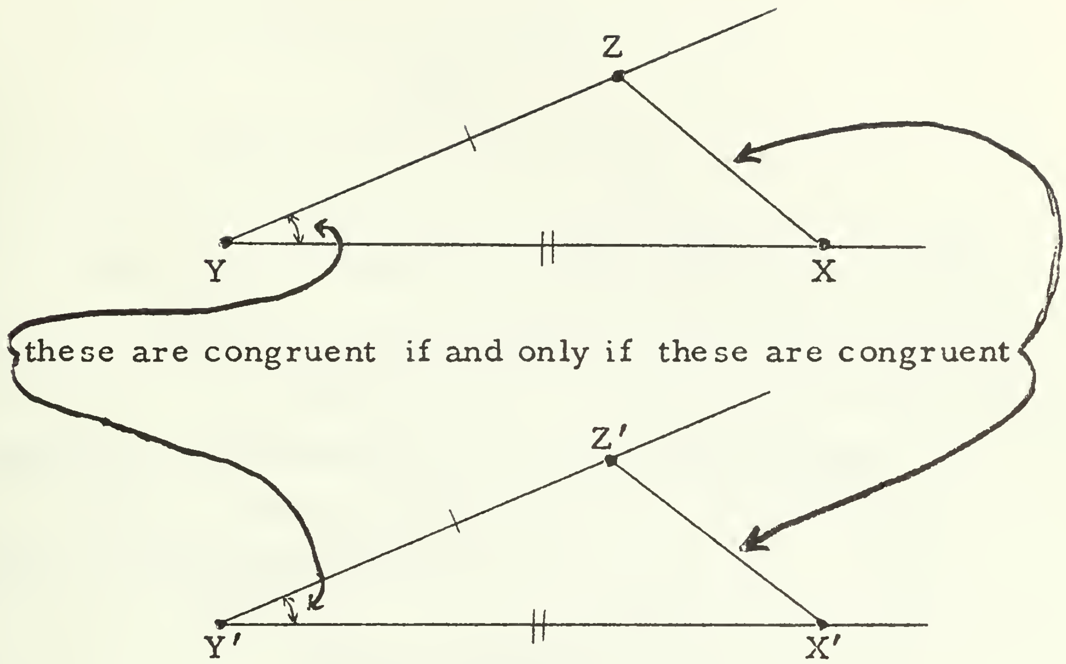
3. Repeat Exercise 1, but now draw two angles with <u>different</u> measure

THE SEGMENT-ANGLE AXIOM

Each axiom up to now has talked either about measures of segments or about measures of angles, but not both. Here is an axiom which is suggested by the preceding exercises, and which does talk about both.

<u>Axiom H.</u>

For each three noncollinear points X, Y, and Z, and each three noncollinear points X′, Y′, and Z′, if XY = X′Y′ and YZ = Y′Z′ then ZX = Z′X′ if and only if $°m(\angle XYZ) = °m(\angle X'Y'Z')$.

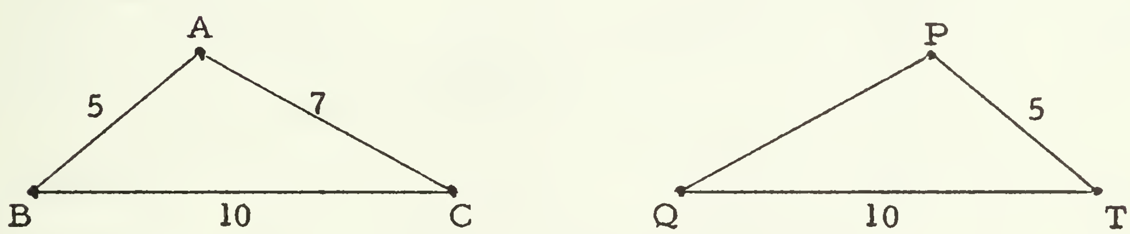


[The slashes indicate the given pairs of congruent sides.]

EXERCISES

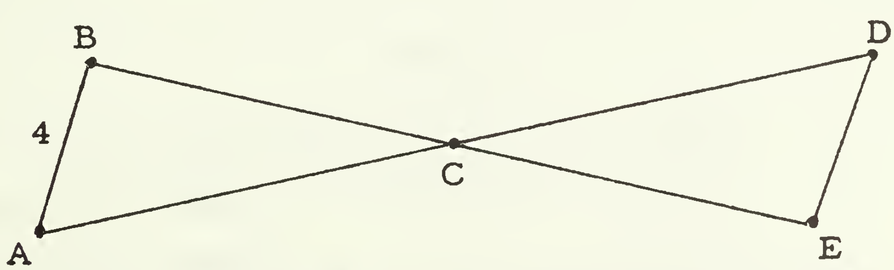
Fill in the blanks and <u>justify</u> your answers.

1.



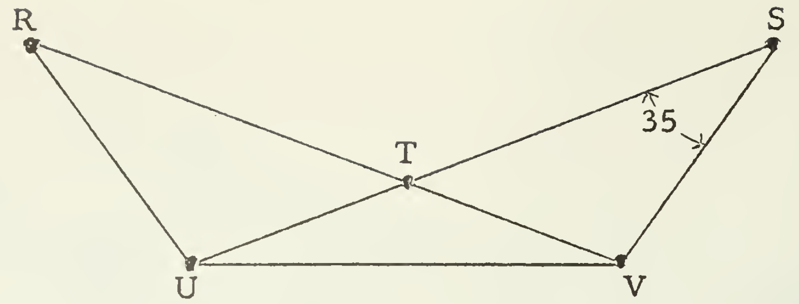
If $\angle B \cong \angle T$ then PQ = ________ .

2.



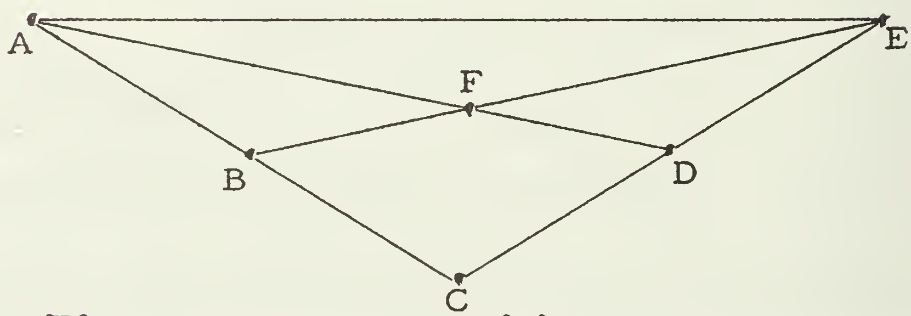
If C is the midpoint of $\overleftrightarrow{AD}$ and C is the midpoint of $\overleftrightarrow{BE}$ then DE = ________ and $\angle A \cong$ ________ .

3.



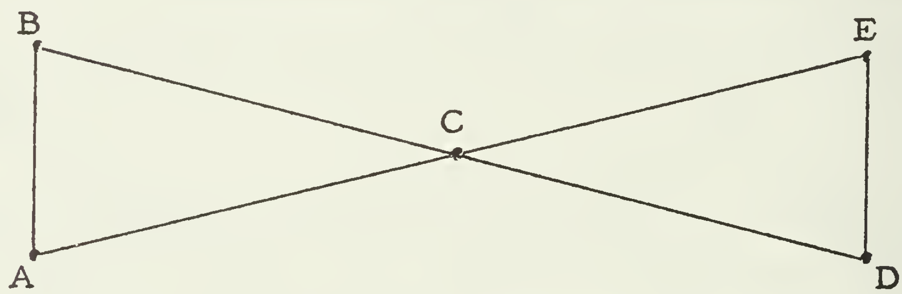
If $\overline{RV} \cong \overline{SU}$ and $\overline{RU} \cong \overline{SV}$ then $m(\angle R) =$ ________ and $\angle SVU \cong$ ________ and $\angle RTS \cong$ ________ .

4.



If $\overline{AC} \cong \overline{EC}$, B is the midpoint of $\overline{AC}$, and D is the midpoint of $\overline{EC}$ then $\overline{AD} \cong$ ________ and $\overline{AB} \cong$ ________ .

5.

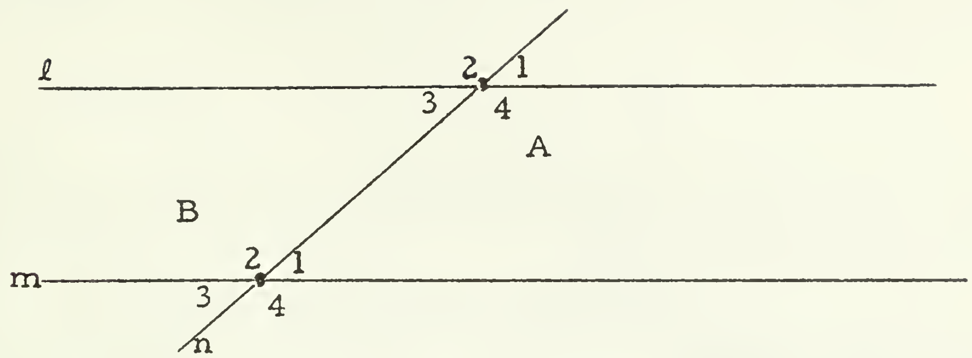


If C is the midpoint of $\overline{BD}$ and of $\overline{AE}$, and $\overline{AE} \cong \overline{BD}$ then $\angle B \cong \angle D$ and $\angle B \cong$ ________ .

6.

If $\overline{AB} \cong \overline{AC}$ then $\angle C \cong$ ______ .
[Hint. For justification, see Exercise 5.]

7.



If $m(\angle A_1) = 40$ and $\angle A_2 \cong \angle B_2$ then
$m(\angle A_2) =$ ___, $m(\angle A_3) =$ ___, $m(\angle A_4) =$ ___, $m(\angle B_1) =$ ___,
$m(\angle B_2) =$ ___, $m(\angle B_3) =$ ___, and $m(\angle B_4) =$ ___.

SUMMARY OF SECTION 6.02

Notation and terminology

acute angle	[6-62]	obtuse angle	[6-62]
adjacent angles	[6-69]	perpendicular lines	[6-67]
angle	[6-51]	right angle	[6-59]
complementary angles	[6-62]	side of an angle	[6-51]
congruent angles	[6-58]	supplementary angles	[6-59]
congruent segments	[6-58]	vertex of an angle	[6-51]
interior of an angle	[6-55]	vertical angles	[6-65]
measure of an angle	[6-51]		
$°m(\angle ABC)$	[6-52]	$\angle ABC$	[6-51]
$m(\angle ABC)$	[6-52]	$\angle ABC \cong \angle D$	[6-58]
$\cong$	[6-58]	$\perp$	[6-67]

Axioms

D. For each three noncollinear points X, Y, and Z

$$0 < °m(\angle XYZ) < 180. \qquad [6\text{-}54]$$

E. For each two points X and Y, each side s of $\overleftrightarrow{XY}$, and each number x such that $0 < x < 180$, there is one and only one half-line h with vertex X and contained in s such that $°m(h \cup \overrightarrow{XY}) = x$. [6-54]

F. For each three noncollinear points X, Y, and Z, and for each point W interior to $\angle XYZ$,

$$°m(\angle XYW) + °m(\angle WYZ) = °m(\angle XYZ).$$ [6-56]

G. For each three noncollinear points X, W, and Z, and for each point $Y \in \overline{XZ}$,

$$°m(\angle XYW) + °m(\angle WYZ) = 180.$$ [6-56]

H. For each three noncollinear points X, Y, and Z, and each three noncollinear points X', Y', and Z', if $XY = X'Y'$ and $YZ = Y'Z'$ then $ZX = Z'X'$ if and only if $°m(\angle XYZ) = °m(\angle X'Y'Z')$. [6-74]

Theorems

2-1. An angle is a right angle if and only if it is an angle of 90°.

2-2. All right angles are congruent.

2-3. Supplements of the same angle or of congruent angles are congruent.

2-4. Complements of the same angle or of congruent angles are congruent.

2-5. Vertical angles are congruent.

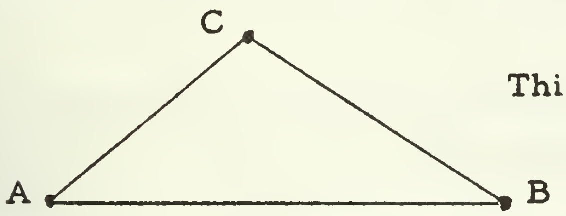
2-6. If two supplementary angles are congruent, they are right angles.

2-7. Each of the four angles contained in the union of two perpendicular lines is a right angle.

2-8. At each point on a line, there is one and only one perpendicular.

2-9. Adjacent angles are supplementary if and only if their noncommon sides are collinear.

6.03 Triangles. -- A triangle is the union of three segments whose end points are three noncollinear points.



This triangle is $\overline{AB} \cup \overline{BC} \cup \overline{CA}$.

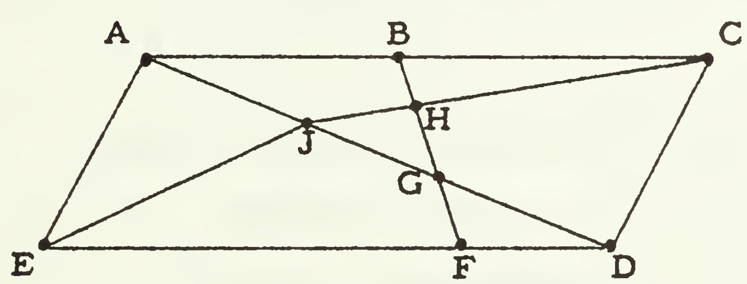
Each of the three noncollinear points is <u>a vertex of the triangle</u>, and each of the three segments is <u>a side of the triangle</u>. Each of the angles which contains two sides of the triangle is <u>an angle of the triangle</u>. [Note that an angle of a triangle is not a subset of the triangle. Explain. How many angles does a triangle have?]

A common way of naming a triangle is to write the names of its three vertices preceded by a 'Δ'. So, $\overline{AB} \cup \overline{BC} \cup \overline{CA}$ is called 'ΔABC' [read this as 'triangle ABC'], or 'ΔBCA', or 'ΔCBA', etc.

For ΔABC shown above, the sides are $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, and the angles are $\angle CAB$, $\angle ABC$, and $\angle BCA$. We say that the side $\overline{BC}$ is the side of ΔABC <u>opposite</u> the vertex A, or opposite $\angle CAB$. Also, $\angle CAB$ is opposite $\overline{BC}$. Which angle of ΔABC is opposite $\overline{CA}$? Which side is opposite $\angle C$?

EXERCISES

1. Name all the triangles shown in the following picture:



2. Draw ΔMNR. Now, draw ΔRST such that $T \in \overline{RM}$ and $N \in \overline{RS}$. Does it follow that there is an angle of ΔMNR which is congruent to an angle of ΔRST?

3. Draw two triangles such that the vertices of one belong to the sides of the other. [Can you draw two triangles such that the vertices of <u>each</u> belong to the sides of the other?]

4. Draw $\triangle RST$ such that $\angle T$ is an acute angle. Mark a point K on the side opposite $\angle S$ such that the perpendicular to the side at K intersects the side opposite $\angle R$.

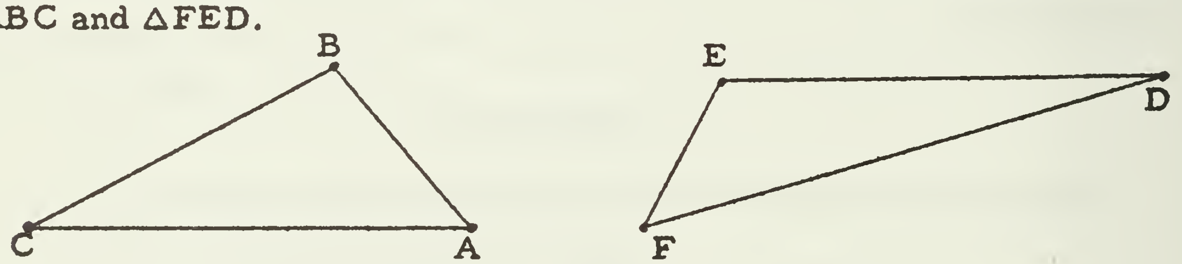
5. Draw $\triangle POQ$ and $\triangle VOU$ such that O is the midpoint of $\overline{VP}$ and O is the midpoint of $\overline{UQ}$. Prove that $\overline{UV} \cong \overline{QP}$.

6. Fill in the blanks. [Can you do it without a picture?]
   In $\triangle TKP$, side $\overline{PK}$ is opposite ______,
   $\angle TPK$ is opposite ______,
   and side $\overline{TK}$ is opposite ______.

CORRESPONDING PARTS OF TRIANGLES

Later we shall say that two triangles are congruent if and only if the vertices of one can be matched with the vertices of the other in such a way that corresponding sides are congruent and corresponding angles are congruent. In order to understand this definition, we must first understand the ideas of matched vertices and of corresponding sides and angles.

The vertices of two triangles can be matched in six ways. Consider $\triangle ABC$ and $\triangle FED$.



We can match the vertices like this: $A \longleftrightarrow F$, $B \longleftrightarrow E$, $C \longleftrightarrow D$. For short, we can describe this matching by writing:

$$(1)\ ABC \longleftrightarrow FED$$

[Read this as 'A, B, C matched with F, E, D'.]

We can also match A with F, B with D, and C with E:

$$(2)\ ABC \longleftrightarrow FDE$$

The other four matchings are indicated by:

(3) $ABC \longleftrightarrow EDF$	(4) $ABC \longleftrightarrow EFD$
(5) $ABC \longleftrightarrow DFE$	(6) $ABC \longleftrightarrow DEF$

[Which of these six matchings is also indicated by '$BAC \longleftrightarrow FDE$'?]

If we choose one of these matchings, we can then speak of <u>corresponding sides</u> and of <u>corresponding angles</u>. With respect to a given matching of vertices, corresponding sides are sides whose end points match; and corresponding angles are angles whose vertices match. For example, with respect to the matching $ABC \leftrightarrow EDF$ of the vertices of $\triangle ABC$ and $\triangle FED$, $\overline{AC}$ and $\overline{EF}$ are corresponding sides, and $\angle BCA$ and $\angle DFE$ are corresponding angles.

EXERCISES

1. Draw triangles $\triangle KQG$ and $\triangle JML$, and list all the pairs of corresponding parts with respect to the matching $KQG \leftrightarrow JML$.

2. List the pairs of corresponding parts [sides and angles] of $\triangle KQG$ and $\triangle JML$ with respect to the matching $QKG \leftrightarrow MLJ$.

3. If, with respect to some matching of the vertices of $\triangle ABC$ and $\triangle PQR$, $\overline{AB}$ and $\overline{PQ}$ are corresponding sides, does it follow
   (a) that $\overline{AC}$ and $\overline{PR}$ are corresponding sides;
   (b) that $\angle C$ and $\angle R$ are corresponding angles?

4. List all of the matchings of the vertices of $\triangle ABC$ with those of $\triangle FED$ for which
   (a) $\overline{AB}$ and $\overline{FD}$ are corresponding sides
   (b) $\angle B$ and $\angle F$ are corresponding angles
   (c) $\overline{AC}$ and $\overline{FE}$ are corresponding sides
   and $\overline{CB}$ and $\overline{DF}$ are corresponding sides
   (d) $\angle ABC$ and $\angle FED$ are corresponding angles
   and $\angle BAC$ and $\angle FDE$ are corresponding angles
   (e) $\overline{AB}$ and $\overline{EF}$ are corresponding sides
   and $\angle ABC$ and $\angle FED$ are corresponding angles
   (f) $\overline{AC}$ and $\overline{EF}$ are corresponding sides
   and $\angle ABC$ and $\angle EDF$ are corresponding angles

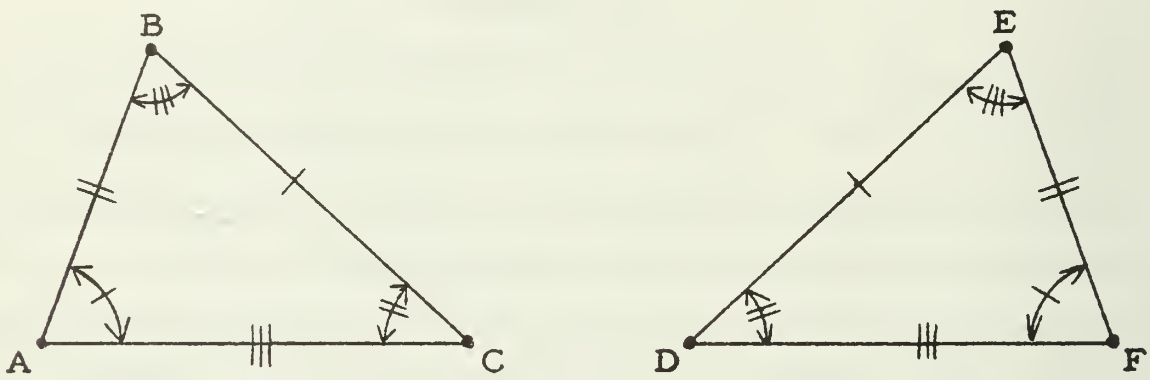
5. The vertices of a triangle can also be matched with themselves in six different ways. For example, for $\triangle ABC$, one such matching is $ABC \leftrightarrow BAC$, Give the other five.

[Supplementary exercises are on page 6-412.]

CONGRUENT TRIANGLES

Suppose that for some matching of the vertices of $\triangle ABC$ with those of $\triangle DEF$, it turns out that all corresponding parts are congruent. In that case, we say that $\triangle ABC$ and $\triangle DEF$ are congruent. In general, a first triangle and a second triangle are <u>congruent</u> if and only if there is a matching of their vertices for which corresponding parts are congruent. Such a matching of their vertices is called <u>a congruence</u>.

Here are two triangles. Are they congruent?



[The slashes tell you that, for example, $\overleftrightarrow{BC} \cong \overleftrightarrow{ED}$ and $\angle B \cong \angle E$.] Now, according to the definition, in order to say that the triangles are congruent, you have to show that some matching of their vertices is a congruence. There are six matchings which you might test. However, it is pretty obvious that if any of them is a congruence, it is $ABC \leftrightarrow FED$. Let's test this matching.

<u>Matching:</u> $ABC \leftrightarrow FED$

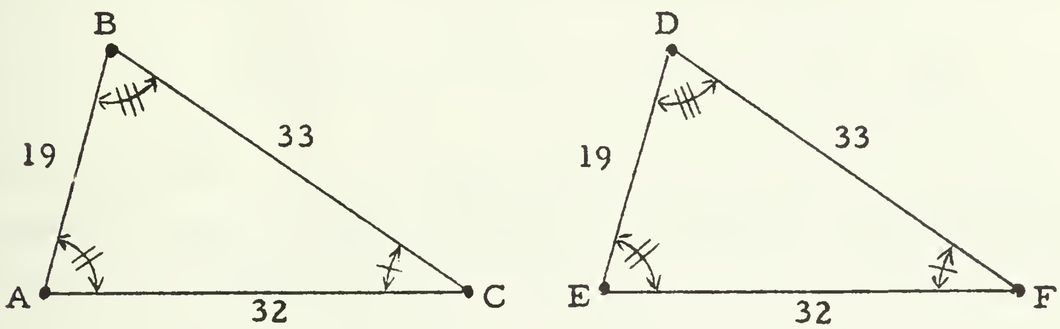
<u>Corresponding parts</u>	<u>Are they congruent?</u>
$\overleftrightarrow{BC}$ and $\overleftrightarrow{ED}$	Yes
$\overleftrightarrow{CA}$ and $\overleftrightarrow{DF}$	Yes
$\overleftrightarrow{AB}$ and $\overleftrightarrow{FE}$	Yes
$\angle A$ and $\angle F$	Yes
$\angle B$ and $\angle E$	Yes
$\angle C$ and $\angle D$	Yes

So, since corresponding parts are congruent, $ABC \leftrightarrow FED$ is a congruence, and, therefore, the triangles are congruent.

EXERCISES

<u>A.</u> Is a triangle congruent to itself? Which of the six matchings of the vertices of $\triangle ABC$ with those of $\triangle ABC$ is a congruence? Can you draw a triangle for which <u>two</u> of these matchings are congruences? More than two?

<u>B.</u>



Consider $\triangle ABC$ and $\triangle DEF$, and the matching $ABC \longleftrightarrow DEF$. With respect to this matching, $\overline{BC}$ and $\overline{EF}$ are corresponding sides. But, since their measures are different, $\overline{BC} \ncong \overline{EF}$. Does it follow that the triangles are not congruent? Explain your answer.

* * *

Suppose you have discovered a congruence for two triangles, or are trying to show that some matching of the vertices is a congruence. In such a case, it is helpful to use names for the triangles which indicate this matching. For example, in Part B above, since $ABC \longleftrightarrow EDF$ is a congruence, it would be convenient to refer to these triangles as '$\triangle ABC$ and $\triangle EDF$' rather than, say, as '$\triangle ABC$ and $\triangle DEF$'. So, although the sentence:

$$(1) \quad \triangle ABC \cong \triangle DEF$$

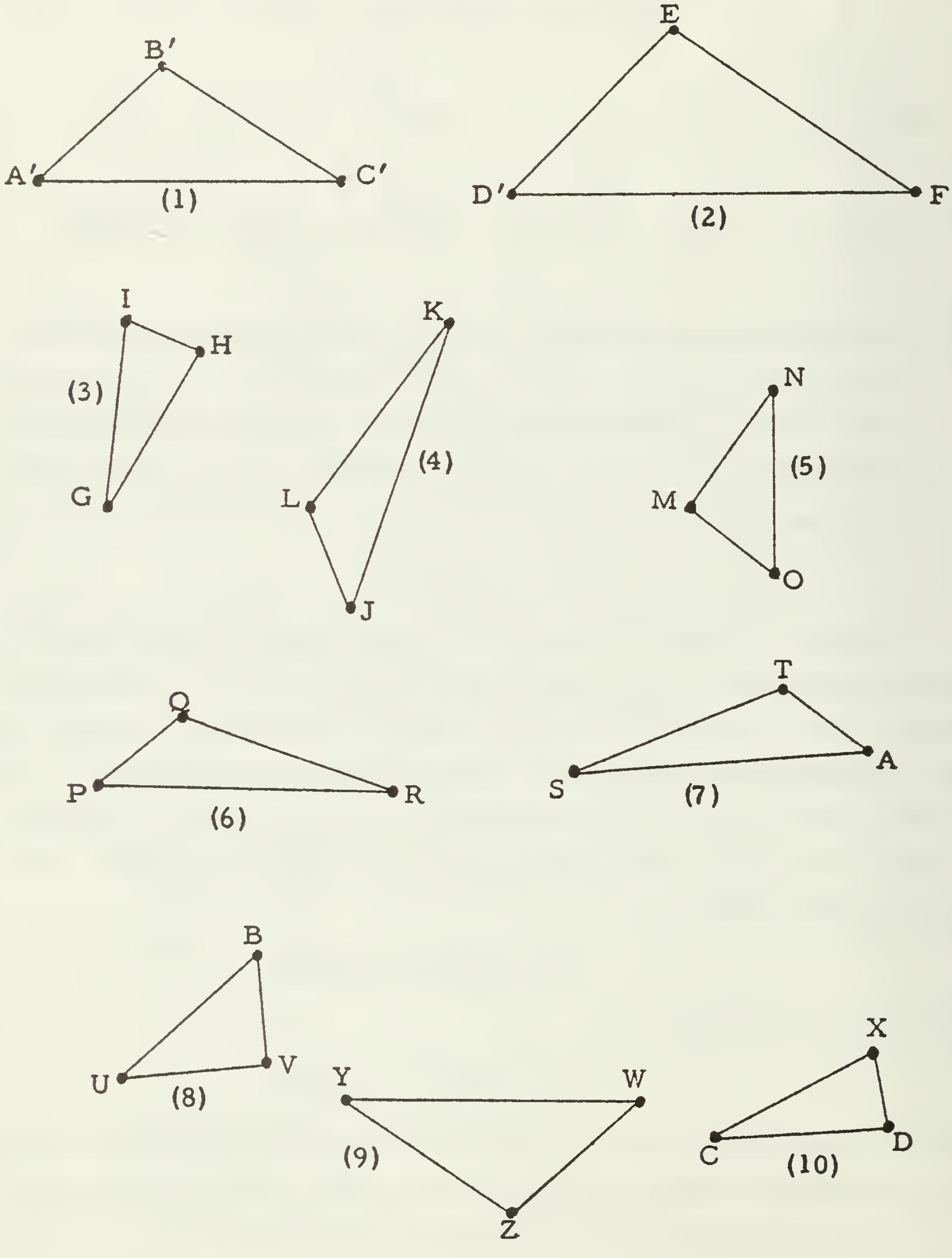
is correct, the sentence:

$$(2) \quad \triangle ABC \cong \triangle EDF$$

is more helpful becuase you can use it to pick out the names of the corresponding congruent parts mechanically. Name the pairs of corresponding parts just by looking at sentence (2).

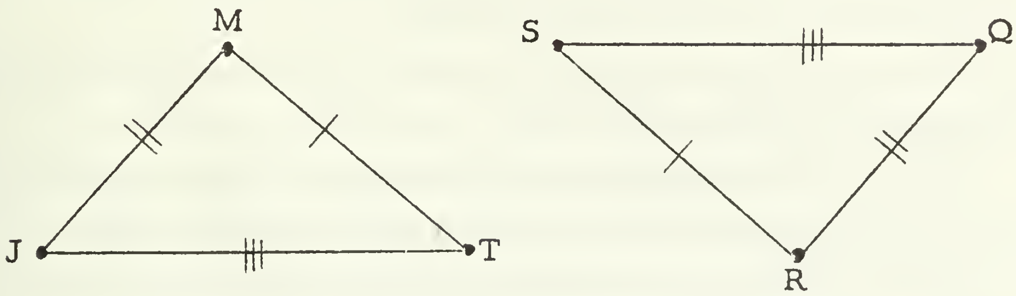
* * *

C. Certain pairs of the triangles pictured below are congruent. Judge by eye which pairs these are, and record your judgments by writing sentences which indicate the congruences of the vertices. [For example, the triangles shown in (3) and (10) are congruent. So, write '$\Delta IHG \cong \Delta XDC$' or '$\Delta GHI \cong \Delta CDX$' rather than '$\Delta IHG \cong \Delta CDX$'.] Also, for each pair of congruent triangles, tell which pairs of angles are congruent, and which pairs of sides are congruent.



TRIANGLE-CONGRUENCE THEOREMS

Here are two triangles. Each side of one triangle is congruent to



a side of the other. Are the triangles, themselves, congruent? According to the definition of congruent triangles, to prove that they are congruent we must show that there is a matching of their vertices for which corresponding parts are congruent--that is, we must show that there is a congruence.

There are six matchings which you might test. However, the matching JTM ⟷ QSR seems most likely to be a congruence. [Why?] Let's test this matching:

<u>Matching:</u> JTM ⟷ QSR

Corresponding parts	Are they congruent?
$\overline{JT}$ and $\overline{QS}$	Yes
$\overline{JM}$ and $\overline{QR}$	Yes
$\overline{TM}$ and $\overline{SR}$	Yes
$\angle J$ and $\angle Q$	?
$\angle T$ and $\angle S$	?
$\angle M$ and $\angle R$	?

Do we have an axiom or a theorem which we might use to establish the congruence of, say, $\angle J$ and $\angle Q$? We can use Axiom H, the segment-angle axiom.

Read Axiom H on page 6-74. Since M, J, and T are noncollinear points and R, Q, and S are noncollinear points, and since $MJ = RQ$ and $JT = QS$, it follows that

$$TM = SR \text{ if and only if } \angle J \cong \angle Q.$$

So, using the only-if-part, since $TM = SR$, we know that $\angle J \cong \angle Q$.

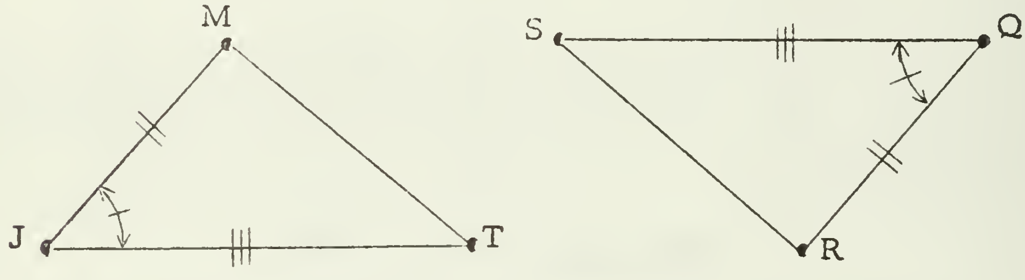
Use Axiom H to show that $\angle T \cong \angle S$. [Since J, T, and M are noncollinear points and . . .] Then, use it to show that $\angle M \cong \angle R$.

So, since $JTM \longleftrightarrow QSR$ is a congruence, it follows that $\triangle JTM \cong \triangle QSR$. This proves the following theorem:

<u>Theorem 3-1.</u> ["s. s. s."]

If, for some matching of the vertices of one triangle with those of a second, each side of the first triangle is congruent to the corresponding side of the second, then this matching is a congruence.

In proving Theorem 3-1 we have used just the only-if-part of Axiom H. This axiom also tells us that if M, J, and T are noncollinear points,



and R, Q, and S are noncollinear points, and if MJ = RQ and JT = QS, then

$$TM = SR \text{ if } \angle J \cong \angle Q.$$

So, if we are given that MJ = RQ, JT = QS, and $\angle J \cong \angle Q$, it follows that TM = SR. But, in that case, $JTM \longleftrightarrow QSR$ is a matching for which each side of $\triangle JTM$ is congruent to the corresponding side of $\triangle QSR$. Hence, by Theorem 3-1, $JTM \longleftrightarrow QSR$ is a congruence. This proves the following theorem:

<u>Theorem 3-2.</u> ["s. a. s."]

If, for some matching of the vertices of one triangle with those of a second, each of two sides of one triangle is congruent to the corresponding side of the second, and the included angles are congruent, then this matching is a congruence.

Note the phrase 'included angle'. The <u>included angle</u> of two sides of a triangle is that angle of the triangle whose vertex is the common end point of the two sides. Similarly, the <u>included side</u> of two angles of a triangle is the side whose end points are the vertices of the two angles.

(1) Without drawing a picture of $\triangle RTQ$, name the angle which is included between the sides $\overline{TQ}$ and $\overline{RQ}$.

(2) Next, name the sides which include $\angle T$.

(3) Name the side included between $\angle R$ and $\angle RTQ$.

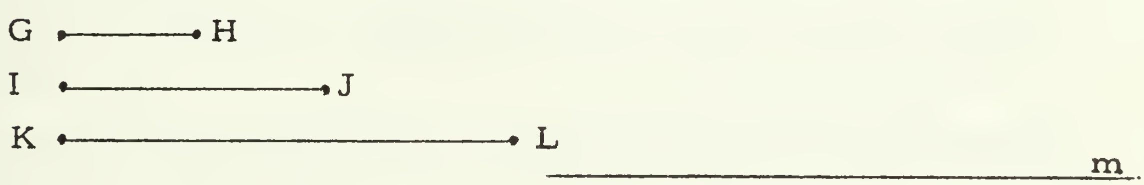
(4) Name the angles which include the side $\overline{TR}$.

EXERCISES

<u>A.</u> 1. Use your compass and straight-edge to draw a triangle whose sides are congruent to the segments pictured below and whose longest side is a subset of a line ℓ.

Can you draw more than one such triangle? Are all such triangles congruent?

2. Repeat Exercise 1 for these three segments and a line m.



3. Mark three noncollinear points A, B, and C. Then,

(a) $\overline{AB} \cup \overline{BC} \cup \overline{CA}$ is a ___________ ;

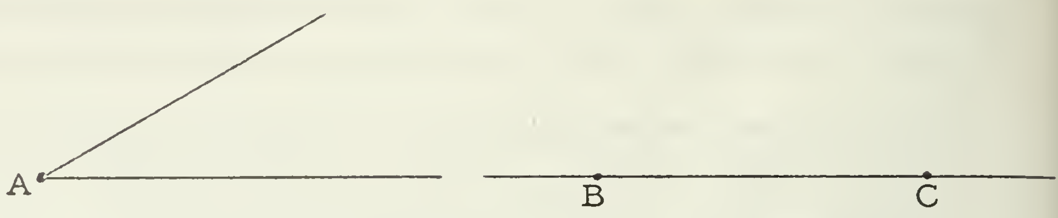
(b) $AB + BC >$ _____ ;

(c) $BC +$ _____ $> AB$;

(d) _____ $+ CA > BC$.

(e) What do you conclude about the sum of the measures of two sides of a triangle? Justify your conclusion.

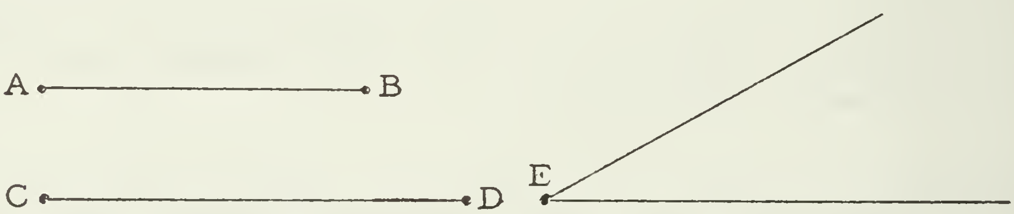
B. Axioms D and E tell us that there is contained in either of the half-



planes determined by $\overleftrightarrow{BC}$ a unique half-line h with vertex B such that $h \cup \overrightarrow{BC} \cong \angle A$. One way to draw this half-line is to use a protractor and a straight-edge.

The s.s.s. theorem suggests another way to draw this half-line, a way which allows you to use a compass in place of the protractor. Figure out this way of drawing an angle which has vertex B, side $\overrightarrow{BC}$, and is congruent to $\angle A$.

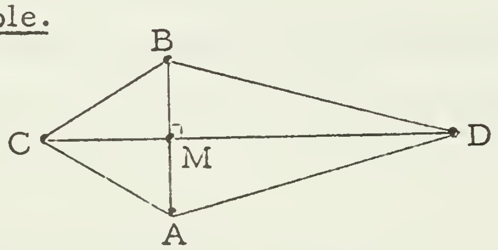
C. Use your compass and straight-edge to draw a triangle with two sides congruent to the given segments and the included angle congruent to the given angle.



[Supplementary exercises are on page 6-414.]

THE TRIANGLE-CONGRUENCE THEOREMS IN PROOFS

Example.



Hypothesis: M is the midpoint of $\overline{AB}$, $\overline{CD} \perp \overline{AB}$ at M

Conclusion: $\triangle ACD \cong \triangle BCD$

Plan. Since $\overline{CD} \cong \overline{CD}$, we can show that $\triangle ACD \cong \triangle BCD$ if we can show that each of the other sides of $\triangle ACD$ is congruent with one of the other sides of $\triangle BCD$, or if we can find a second pair of congruent sides such that the angles included between each of them and $\overline{CD}$ are congruent.

It looks as though $\overline{AC} \cong \overline{BC}$, and if we can prove this, the same kind of proof would show that $\overline{AD} \cong \overline{BD}$. So, by Theorem 3-1, we would know that ACD $\longleftrightarrow$ BCD is a congruence. Now, $\overline{AC}$ and $\overline{BC}$ appear to be corresponding sides of congruent triangles [the small ones]. So, let's try to prove that <u>these</u> triangles are congruent. Now that we have an idea of how to derive the conclusion from the hypothesis, let's do so.

(1)	A, M, C and B, M, C are vertices of triangles	[figure]
(2)	$\overline{AM} \cong \overline{MB}$	[Hypothesis; def. of midpoint]
(3)	$\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$ at M	[Hypothesis]
(4)	Each of the four angles contained in the union of two perpendicular lines is a right angle.	[Theorem 2-7]
(5)	$\angle$AMC and $\angle$BMC are right angles	[(3) and (4)]
(6)	All right angles are congruent.	[Theorem 2-2]
(7)	$\angle AMC \cong \angle BMC$	[(5) and (6)]
(8)	$\overline{MC} \cong \overline{MC}$	[Identity; def. of congruent segments]
(9)	s.a.s.	[Theorem 3-2]
(10)	AMC $\longleftrightarrow$ BMC is a congruence	[(1), (2), (7), (8), and (9)]
(11)	$\overline{AC} \cong \overline{BC}$	[(10); def. of congruence]

[In the plan, we suggested going on from here to prove that $\overline{AD} \cong \overline{BD}$. We could do this, and then reach our desired conclusion with the help of s.s.s. But, let's continue with another approach.]

(12)	A, C, D and B, C, D are vertices of triangles	[figure]
(13)	$\angle ACD \cong \angle BCD$	[(10); def. of congruence]
(14)	$\overline{CD} \cong \overline{CD}$	[Identity; def. of congruent segments]
(15)	ACD $\longleftrightarrow$ BCD is a congruence	[(12), (11), (13), (14), and (9)]
(16)	$\Delta ACD \cong \Delta BCD$	[(15); def. of congruent triangles]

A few remarks are in order about the preceding proof.

Look at (8). By the definition of congruent segments, '$\overline{MC} \cong \overline{MC}$' is just another way of saying 'MC = MC'. This last is an instance of the logical principle of identity. So, we write '[Identity; def. of congruent segments]' as an explanation of step (8).

We have, as explained earlier, omitted instances of theorems.

Finally, instead of stating Theorem 3-2 in step (9), we have given its common name.

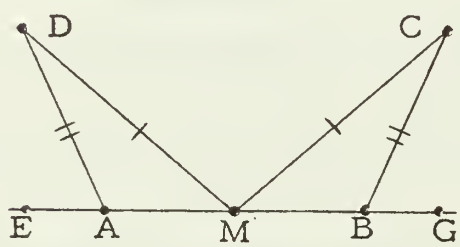
*

As usual, a paragraph proof is shorter. Here is one which is inspired by the original plan:

> Since M is the midpoint of $\overline{AB}$, $\overline{AM} \cong \overline{MB}$. Since $\overline{CD} \perp \overline{AB}$ at M, $\angle AMC$ and $\angle BMC$ are right angles and, so, are congruent. Also, $\overline{MC} \cong \overline{MC}$. Hence, by s.a.s., $AMC \longleftrightarrow BMC$ is a congruence. So, $\overline{AC} \cong \overline{BC}$. Similarly, $\overline{AD} \cong \overline{BD}$. Since, also, $\overline{CD} \cong \overline{CD}$, it follows by s.s.s. that $ACD \longleftrightarrow BCD$ is a congruence. So, $\triangle ACD \cong \triangle BCD$.

EXERCISES

A. 1.

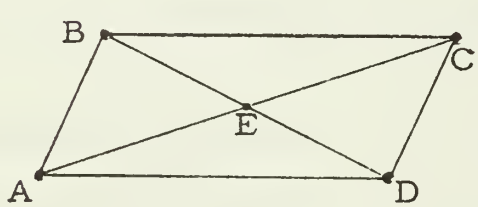


Hypothesis: M is the midpoint of $\overline{AB}$, $\overline{MC} \cong \overline{MD}$, $\overline{BC} \cong \overline{AD}$

Conclusion: $\angle D \cong \angle C$, $\angle DAE \cong \angle CBG$

[Hint. First show that $AMD \longleftrightarrow BMC$ is a congruence.]

2.



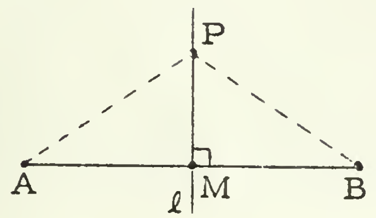
Hypothesis: $\overline{AC}$ and $\overline{BD}$ bisect each other at E

Conclusion: $\overline{AB} \cong \overline{DC}$, $\overline{BC} \cong \overline{DA}$

[Note. We say that a set bisects a segment if and only if their intersection consists of just the midpoint of the segment.

[Supplementary exercises are on page 6-417.]

B. Complete the column proof.



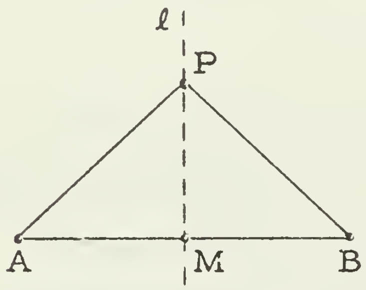
Hypothesis: $\ell \perp \overleftrightarrow{AB}$ at M,
M is the midpoint of $\overline{AB}$,
$P \in \ell$

Conclusion: P is equidistant from A and B
[that is, $\overline{PA} \cong \overline{PB}$]

(1)	$P \in \ell$	[________]
(2)	$\ell \perp \overleftrightarrow{AB}$ at M	[________]
(3)	$\ell \cap \overleftrightarrow{AB} = \{M\}$	[(2); def. of $\perp$]
(4)	$P = M$ or $P \notin \overleftrightarrow{AB}$	[(1) and (3)]
(5)	$P = M$	[assumption]
(6)	________	[Hypothesis]
(7)	________	[(6); def. of midpoint]
(8)	________	[(5) and (7)]
(9)	if $P = M$ then P is equidistant from A and B	[____ ; ________]
(10)	$P \notin \overleftrightarrow{AB}$	[assumption]
(11)	P, M, A and P, M, B are vertices of triangles	[(10); Hypothesis]
(12)	$\overline{PM} \cong \overline{PM}$	[____ ; ________]
(13)	________	[Theorem 2-7]
(14)	________	[(2) and (13)]
(15)	________	[Theorem 2-2]
(16)	________	[(14) and (15)]
(17)	$\overline{MA} \cong \overline{MB}$	[(6); def. of midpoint]
(18)	s.a.s.	[Theorem 3-2]
(19)	$PMA \longleftrightarrow PMB$ is a congruence	[________]
(20)	P is equidistant from A and B	[(19); def. of congruence]
(21)	if $P \notin \overleftrightarrow{AB}$ then P is equidistant from A and B	[____ ; † ______]
(22)	P is equidistant from A and B	[(4), (9), and (21)]

[The rule of reasoning which explains the last step is discussed in the Appendix on page 6-393.]

C. In Part B you proved that, given a nondegenerate segment, each point on the line which is perpendicular to the segment at its midpoint is equidistant from the end points of the segment. Now, let's consider the converse. Given a nondegenerate segment $\overline{AB}$, is it the case that each point P which is equidistant from A and B belongs to the line perpendicular to $\overline{AB}$ at its midpoint?



Hypothesis: $\ell \perp \overleftrightarrow{AB}$ at M,
M is the midpoint of $\overline{AB}$,
$\overline{PA} \cong \overline{PB}$

Conclusion: $P \in \ell$

Complete the following column proof.

(1)	$P \in \overleftrightarrow{AB}$ or $P \notin \overleftrightarrow{AB}$	[principle of logic]
(2)	$P \in \overleftrightarrow{AB}$	[assumption]
(3)	$\overline{PA} \cong \overline{PB}$	[____________]
(4)	________________	[Theorem 1-9]
(5)	________________	[(2), (3), and (4)]
(6)	M is the midpoint of $\overline{AB}$	[____________]
(7)	________________	[(5) and (6)]
(8)	$M \in \ell$	[Hypothesis]
(9)	________________	[(7) and (8)]
(10)	if $P \in \overleftrightarrow{AB}$ then $P \in \ell$	[(9); * ________]
(11)	$P \notin \overleftrightarrow{AB}$	[assumption]
(12)	P, M, A and P, M, B are vertices of triangle	[____________]
(13)	$\overline{PA} \cong \overline{PB}$	[____________]
(14)	________________	[(6); def. of midpoint]
(15)	________________	[___ ; ________]
(16)	s. s. s.	[Theorem 3-1]
(17)	________________	[(12), (13), (14), (15), and (16)]
(18)	$\angle PMA \cong \angle PMB$	[___ ; ________]

(19)	$\angle PMA$ and $\angle PMB$ are adjacent angles whose non-common sides are collinear	[(6) and (11); def. of adjacent angles]
(20)	____________________	[Theorem 2-9]
(21)	____________________	[(19) and the ___ -part of (20)]
(22)	____________________	[Theorem 2-6]
(23)	____________________	[(18), (21), and (22)]
(24)	$\overleftrightarrow{PM} \perp \overleftrightarrow{AB}$ at M	[(23); def. of perpendicular lines]
(25)	$\ell \perp \overleftrightarrow{AB}$ at M	[______________]
(26)	____________________	[Theorem 2-8]
(27)	$\overleftrightarrow{PM} = \ell$	[(24), (25), and (26)]
(28)	$P \in \ell$	[(27)]
(29)	if $P \notin \overleftrightarrow{AB}$ then $P \in \ell$	[(28); † ________]
(30)	$P \in \ell$	[(1), (10), and (29)]

* * *

D. <u>The</u> <u>interior</u> <u>of</u> <u>a</u> <u>triangle</u> is the intersection of the interiors of the three angles of the triangle. So, a point is in the interior of a triangle if and only if it belongs to the interior of each of the angles of the triangle.

1. Draw $\triangle ABC$. Shade the interior of the triangle. Can a point belong to the intersection of the interiors of two angles of the triangle and not belong to the interior of the triangle?

2. Draw $\triangle JKL$. Locate a point D in the interior of this triangle and a point E on $\overrightarrow{JK}$ such that $\triangle JKL \cong \triangle JDE$.

E. <u>The</u> <u>exterior</u> <u>of</u> <u>a</u> <u>triangle</u> is the complement of the union of the triangle and its interior. So, a point belongs to the exterior of a triangle if and only if it does not belong to the triangle and it does not belong to the interior of the triangle.

1. Draw $\triangle ABC$. Can you locate a point D in the exterior of $\triangle ABC$ and a point E in the interior of $\triangle ABC$ such that $\overline{DE} \cap \triangle ABC = \emptyset$?

2. Draw $\triangle RST$. Locate a point M in the exterior of this triangle and a point N on $\overrightarrow{ST}$ such that $\triangle MRS \cong \triangle MNS$.

PERPENDICULAR BISECTOR

The work in Parts B and C on pages 6-91 and 6-92 shows that a point is equidistant from the two end points of a segment if and only if it belongs to a line which is perpendicular to the segment at its midpoint. This line is called <u>the perpendicular bisector of the segment</u>. So, we have the following theorem:

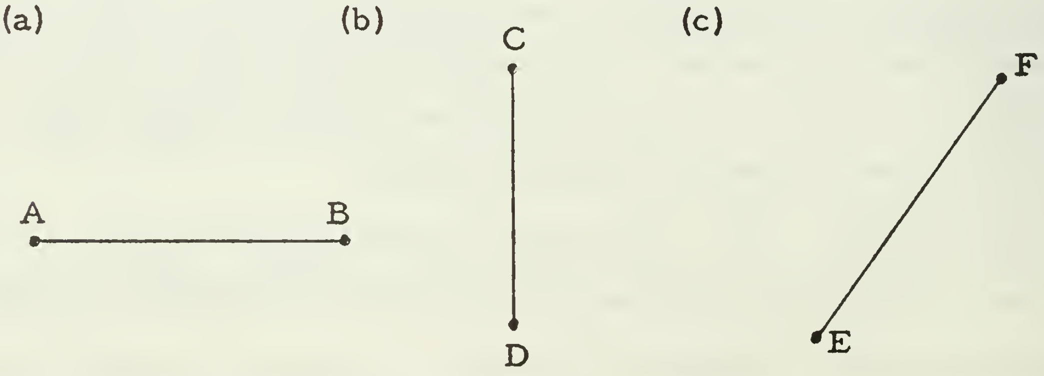
> <u>Theorem 3-3.</u>
> A point is equidistant from the two end points of a segment if and only if it belongs to the perpendicular bisector of the segment.

Theorem 3-3 is helpful in drawing the perpendicular bisector of a segment. Of course, one way to do this is to use a ruler to find the midpoint of the segment, and then use a protractor to find another point on the perpendicular bisector. [Explain how this is done.] But, Theorem 3-3 suggests a method for drawing the perpendicular bisector using only straight-edge and compass. Just find any two points equidistant from the end points of the segment. These two points determine the line which is the perpendicular bisector. Use this idea in doing the exercises which follow.

EXERCISES

<u>A.</u> 1. Draw the perpendicular bisector of each of the given segments using straight-edge and compass only.

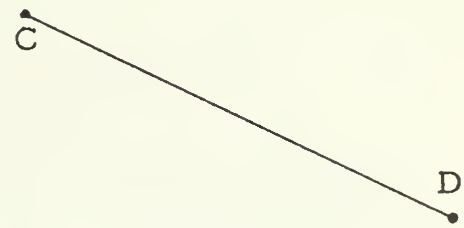
(a) (b) (c)



2. Use straight-edge and compass only to find the midpoints of these segments.

(a)

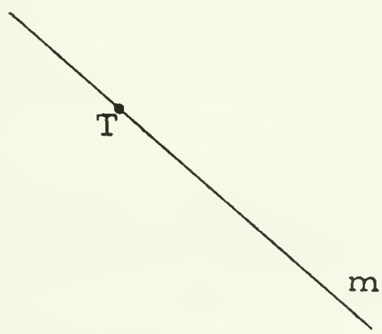
(b)



3. Use straight-edge and compass only to draw the perpendicular to each of the given lines at the given points.

(a)

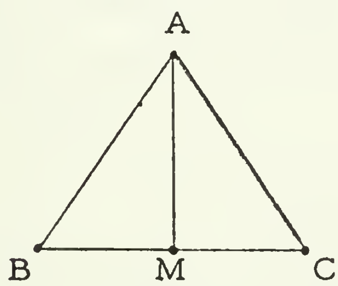
(b)



(c)

B. In each of the following exercises tell why the conclusion follows from the hypothesis. [You needn't write column proofs.]

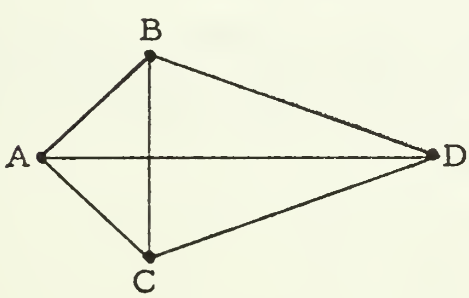
1.



Hypothesis: $\overline{AB} \cong \overline{AC}$,
M is the midpoint of $\overline{BC}$

Conclusion: $\overline{AM} \perp \overline{BC}$ at M

2.



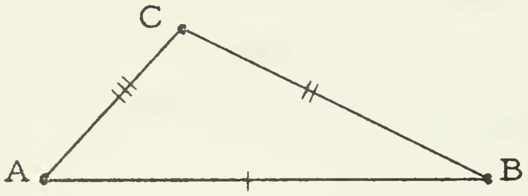
Hypothesis: $\overline{AB} \cong \overline{AC}$,
$\overline{DB} \cong \overline{DC}$

Conclusion: $\overline{BC} \perp \overline{AD}$

[Supplementary exercises are on page 6-418.]

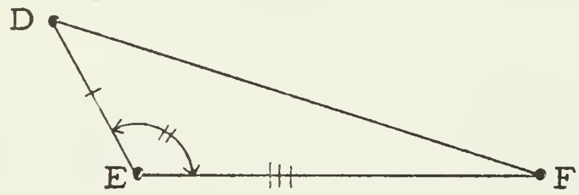
EXPLORATION EXERCISES

A. 1. Here is a picture of $\triangle ABC$. Draw a picture of another triangle, $\triangle A'B'C'$, such that $A'B' = AB$, $B'C' = BC$, and $C'A' = CA$.



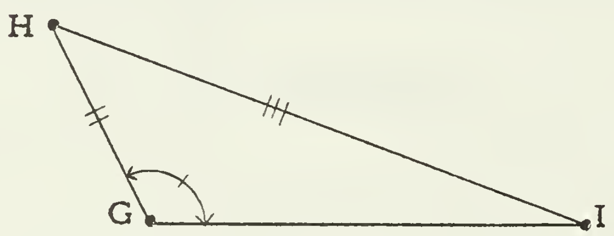
2. Is $\triangle A'B'C' \cong \triangle ABC$? Could you have carried out the instructions and drawn a triangle not congruent to $\triangle ABC$?

B. 1. Here is a picture of $\triangle DEF$. Draw a picture of another triangle, $\triangle D'E'F'$, such that $D'E' = DE$, $\angle E' \cong \angle E$, and $E'F' = EF$.



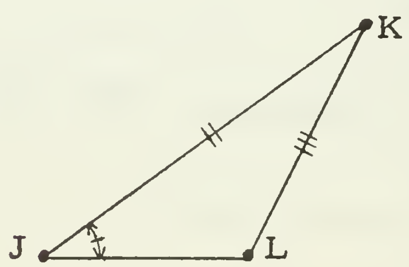
2. Is $\triangle D'E'F' \cong \triangle DEF$? Could you have carried out the instructions and drawn a triangle not congruent to $\triangle DEF$?

C. 1. Here is a picture of $\triangle GHI$. Draw a picture of another triangle, $\triangle G'H'I'$, such that $\angle G' \cong \angle G$, $G'H' = GH$, and $H'I' = HI$.



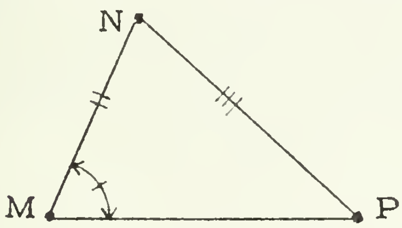
2. Is $\triangle G'H'I' \cong \triangle GHI$? Could you have carried out the instructions and drawn a triangle not congruent to $\triangle DEF$?

D. 1. Here is a picture of $\triangle JKL$. Draw a picture of another triangle, $\triangle J'K'L'$, such that $\angle J' \cong \angle J$, $J'K' = JK$, and $K'L' = KL$.



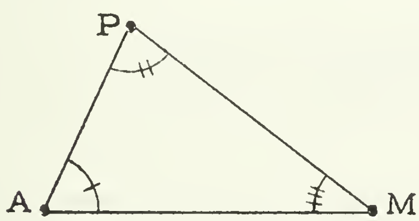
2. Is $\Delta J'K'L' \cong \Delta JKL$? Could you have carried out the instructions and drawn a triangle not congruent to ΔJKL?

E. 1. Here is a picture of ΔMNP. Draw a picture of another triangle, $\Delta M'N'P'$, such that $\angle M' \cong \angle M$, $M'N' = MN$, and $N'P' = NP$.



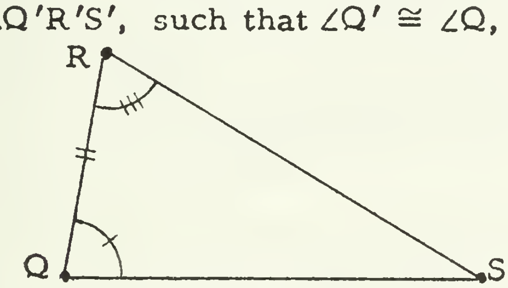
2. Is $\Delta M'N'P' \cong \Delta MNP$? Could you have carried out the instructions and drawn a triangle not congruent to ΔMNP?

F. 1. Here is a picture of ΔAMP. Draw a picture of another triangle, $\Delta A'M'P'$, such that $\angle A' \cong \angle A$, $\angle M' \cong \angle M$, and $\angle P' \cong \angle P$.



2. Is $\Delta A'M'P' \cong \Delta AMP$? Could you have carried out the instructions and drawn a triangle not congruent to ΔAMP?

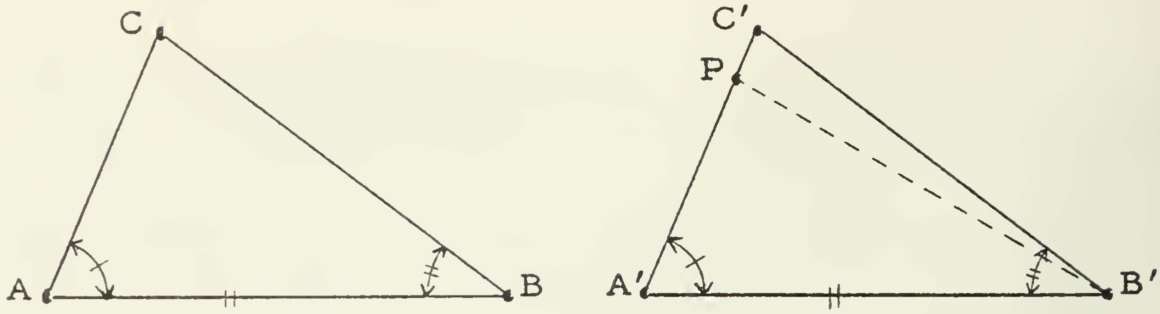
G. 1. Here is a picture of ΔQRS. Draw a picture of another triangle, $\Delta Q'R'S'$, such that $\angle Q' \cong \angle Q$, $Q'R' = QR$, and $\angle R' \cong \angle R$.



2. Is $\Delta Q'R'S' \cong \Delta QRS$? Could you have carried out the instructions and drawn a triangle not congruent to ΔQRS?

ANOTHER TRIANGLE-CONGRUENCE THEOREM

We have already shown that two triangles are congruent if they "agree" in their sides or if they agree in two sides and the included angle. Part G of the preceding Exploration Exercises may have suggested to you that two triangles are congruent if they agree in two angles and the included side. This is the case. Let's prove it.



Suppose that $\angle A' \cong \angle A$, that $A'B' = AB$, and that $\angle A'B'C' \cong \angle B$. By Axiom C, there is one and only one point P such that P belongs to the half-line $\overrightarrow{A'C'}$ and $A'P = AC$. By s.a.s., it follows that

$$(*) \quad PA'B' \longleftrightarrow CAB \text{ is a congruence.}$$

Now, by the definition of a congruence, $\angle A'B'P \cong \angle B$. Since, by assumption, $\angle A'B'C'$ is also congruent to $\angle B$, it follows that $\angle A'B'P \cong \angle A'B'C'$. So, by Axioms D and E, we conclude that the half-lines $\overrightarrow{B'C'}$ and $\overrightarrow{B'P}$ are the same. Since $P \in \overrightarrow{A'C'}$ and $P \in \overrightarrow{B'C'}$, it follows that $P = C'$. [Explain.]

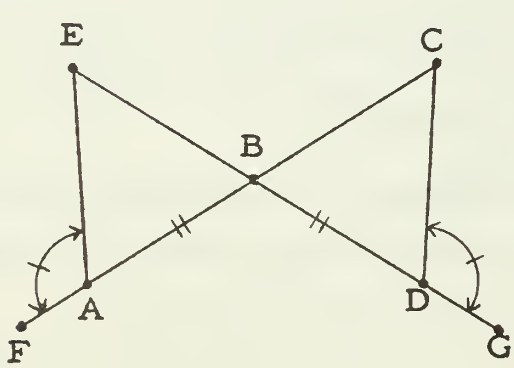
Hence, from this last result and (*), we see that

$$C'A'B' \longleftrightarrow CAB \text{ is a congruence.}$$

Thus, we have our third triangle-congruence theorem:

> Theorem 3-4. ["a.s.a."]
>
> If, for some matching of the vertices of one triangle with those of a second, each of two angles of one triangle is congruent to the corresponding angle of the second, and the included sides are congruent, then this matching is a congruence.

Example.



Hypothesis: $\overleftrightarrow{AB} \cong \overleftrightarrow{DB}$

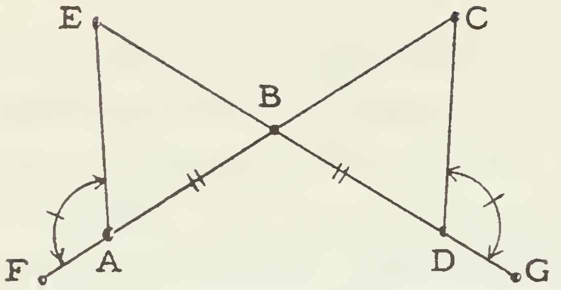
$\angle EAF \cong \angle CDG$

Conclusion: $\overleftrightarrow{AC} \cong \overleftrightarrow{DE}$

<u>Plan.</u> Since $\overleftrightarrow{AB} \cong \overleftrightarrow{DB}$, let's try to show that $\overleftrightarrow{BC} \cong \overleftrightarrow{BE}$. To do this, it is sufficient to show that $ABE \leftrightarrow DBC$ is a congruence. Since $\angle ABE$ and $\angle DBC$ are vertical angles, we see that $\triangle ABE$ and $\triangle DBC$ agree in a pair of angles and a pair of sides. Use either s.a.s. or a.s.a. The first is out because we are trying to show that $\overleftrightarrow{EB} \cong \overleftrightarrow{CB}$. So, try to show that $\angle EAB \cong \angle CDB$. This we can do because they are supplements of angles which are given to be congruent.

<u>Solution I.</u> [column proof]

(1)	$\angle EAF$ and $\angle EAB$ are adjacent angles with their noncommon sides collinear	[figure]
(2)	Adjacent angles are supplementary if and only if their noncommon sides are collinear.	[theorem]
(3)	$\angle EAF$ and $\angle EAB$ are supplementary	[(1) and the if-part of (2)]
(4)	$\angle CDG$ and $\angle CDB$ are supplementary	[steps like (1) and (2)]
(5)	$\angle EAF \cong \angle CDG$	[Hypothesis]
(6)	Supplements of the same angle or of congruent angles are congruent.	[theorem]
(7)	$\angle EAB \cong \angle CDB$	[(3), (4), (5), and (6)]
(8)	$\overleftrightarrow{AB} \cong \overleftrightarrow{DB}$	[Hypothesis]
(9)	$\angle ABE$ and $\angle DBC$ are vertical angles	[figure]
(10)	Vertical angles are congruent.	[theorem]
(11)	$\angle ABE \cong \angle DBC$	[(9) and (10)]
(12)	A, B, E and D, B, C are vertices of triangles	[figure]
(13)	a.s.a.	[theorem]
(14)	$ABE \leftrightarrow DBC$ is a congruence	[(12), (7), (8), (11), and (13)]
(15)	$\overleftrightarrow{BE} \cong \overleftrightarrow{BC}$	[(14); def. of congruence]
(16)	$B \in \overleftrightarrow{AC}$	[figure]
(17)	$\forall_X \forall_Y \forall_Z$ if $Y \in \overleftrightarrow{XZ}$ then $XY + YZ = XZ$	[axiom]
(18)	$AB + BC = AC$	[(16) and (17)]
(19)	$DB + BE = DE$	[steps like (16) and (17)]
(20)	$\overleftrightarrow{AC} \cong \overleftrightarrow{DE}$	[(8), (15), (18), (19); def. of congruent segments]



Hypothesis: $\overline{AB} \cong \overline{DB}$, $\angle EAF \cong \angle CDG$

Conclusion: $\overline{AC} \cong \overline{DE}$

Solution II. [Paragraph proof with footnotes]

From the figure, $A \in \overline{FB}$. So, $\angle EAF$ and $\angle EAB$ are supplementary angles (1). Similarly, $\angle CDG$ and $\angle CDB$ are supplementary angles. So, since by hypothesis, $\angle EAF \cong \angle CDG$, it follows that $\angle EAB \cong \angle CDB$ (2). Also, by hypothesis, $\overline{AB} \cong \overline{DB}$. Now, from the figure, $\angle ABE$ and $\angle DBC$ are vertical angles, and, so, are congruent (3). Hence, for the triangles $\triangle ABE$ and $\triangle DBC$, it follows from the a. s. a. theorem that $ABE \longleftrightarrow DBC$ is a congruence, and from this it follows that $\overline{BE} \cong \overline{BC}$. Now, from the figure, $B \in \overline{AC}$. So, from one of our axioms (4), we know that $AB + BC = AC$. Similarly, $DB + BE = DE$. So, since $AB = DB$ and $BC = BE$, it follows that $AC = DE$, that is, $\overline{AC} \cong \overline{DE}$.

---

(1) Adjacent angles are supplementary if and only if their noncommon sides are collinear.

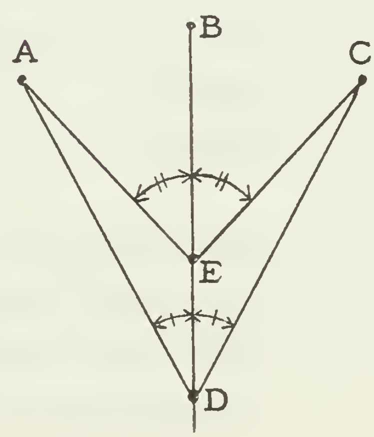
(2) Supplements of the same angle or of congruent angles are congruent.

(3) Vertical angles are congruent.

(4) $\forall_X \forall_Y \forall_Z$ if $Y \in \overline{XZ}$ then $XY + YZ = XZ$

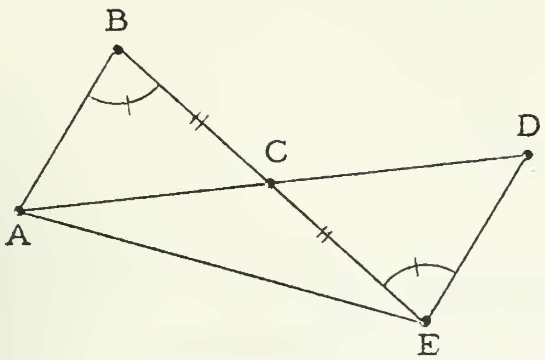
EXERCISES

A. 1.



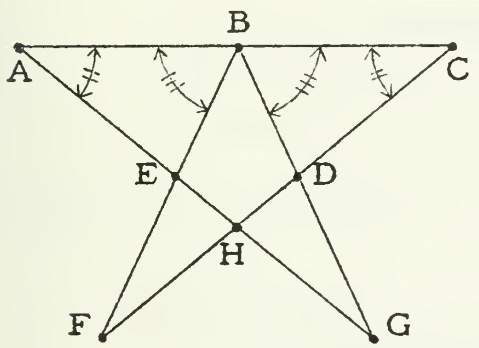
Hypothesis: $\angle AEB \cong \angle CEB$, $\angle ADB \cong \angle CDB$

Conclusion: $\overline{AE} \cong \overline{CE}$

2. 

<u>Hypothesis:</u> $\overline{AD}$ bisects $\overline{BE}$,
$\angle B \cong \angle BED$

<u>Conclusion:</u> $\overline{BE}$ bisects $\overline{AD}$

3. 

<u>Hypothesis:</u> B is the midpoint of $\overline{AC}$,
$\angle A \cong \angle C$,
$\angle ABF \cong \angle CBG$

<u>Conclusion:</u> $\overline{AG} \cong \overline{CF}$

4. Suppose that $\overline{AD} \cap \overline{BC} = \{E\}$, and that $\overline{AB} \cong \overline{CD}$. Also, suppose that $\angle BAD \cong \angle ADC$ and that $\angle ABC \cong \angle BCD$. If F is a point in $\overline{AB}$ and $\overleftrightarrow{FE} \cap \overline{CD} = \{G\}$, show that $\overline{FE} \cong \overline{EG}$.

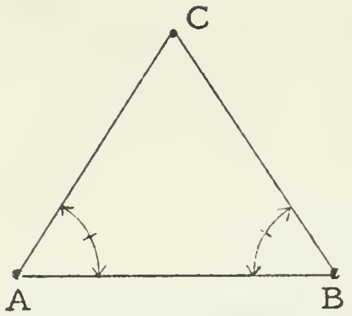
5. Suppose that D is a point in the interior of $\angle ABC$ and that $\angle DBA \cong \angle DBC$. Let ℓ be the line perpendicular to $\overleftrightarrow{BD}$ at D. Suppose that ℓ intersects $\overline{BA}$ and $\overline{BC}$ in M and N, respectively. Show that $\angle DMB \cong \angle DNB$.

6. In earlier exercises [Parts D and E on page 6-93], we defined the interior of a triangle to be the intersection of the interiors of the three angles of the triangle. We also defined the exterior of a triangle to be the complement of the union of the triangle and its interior.

   (a) Draw $\triangle ABC$. Make a sketch showing the intersection of the exteriors of the three angles of $\triangle ABC$. Is this intersection the same as the exterior of the triangle?

   (b) Draw $\triangle ABC$, again. Show, by shading, the set of all points which belong to the interior of none of the angles of $\triangle ABC$. Is this set the same as the one in (a)?

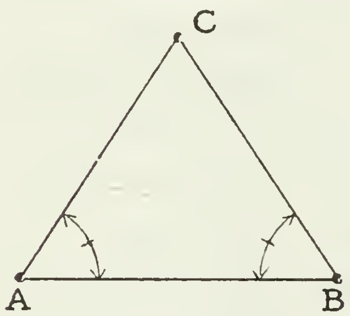
B. 1.



Hypothesis: $\angle A \cong \angle B$

Conclusion: $ABC \longleftrightarrow BAC$ is a congruence

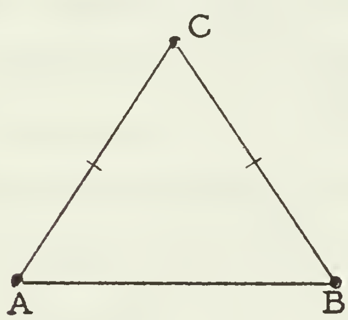
2.



Hypothesis: $\angle A \cong \angle B$

Conclusion: $\overline{BC} \cong \overline{AC}$

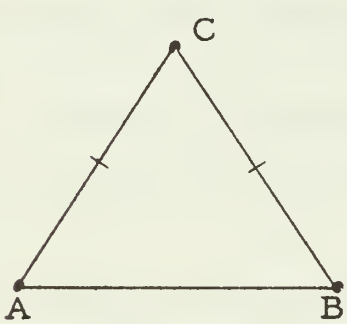
C. 1.



Hypothesis: $\overline{AC} \cong \overline{BC}$

Conclusion: $ACB \longleftrightarrow BCA$ is a congruence

2.



Hypothesis: $\overline{AC} \cong \overline{BC}$

Conclusion: $\angle B \cong \angle A$

[Did you notice that the situation in Exercise 2 of Part B is the "converse" of the situation in Exercise 2 of Part C?]

ISOSCELES AND EQUILATERAL TRIANGLES

A triangle is called <u>an isosceles triangle</u> if and only if two of its sides are congruent. [A triangle which is not an isosceles triangle is sometimes called <u>a scalene triangle</u>.] In Parts B and C on page 6-102 you proved two theorems:

> If two angles of a triangle are congruent then the sides opposite the angles are congruent,

and:

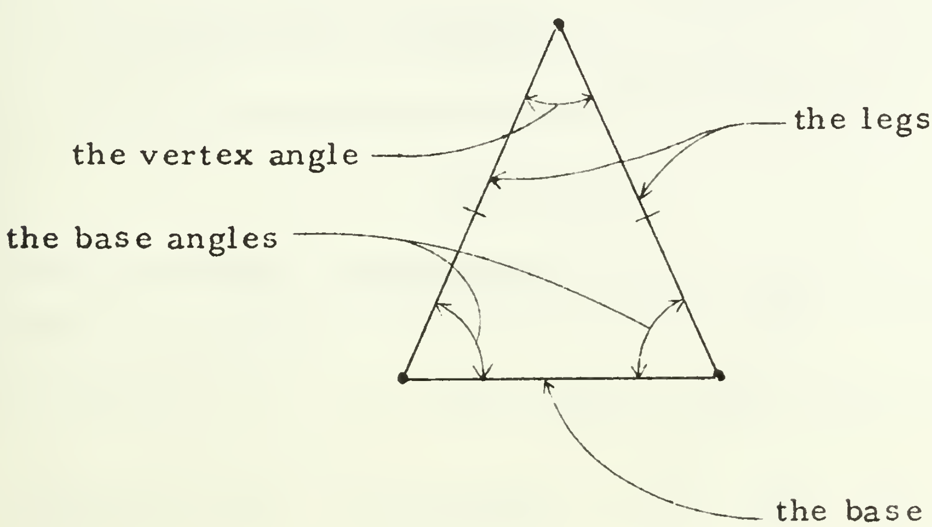
> If two sides of a triangle are congruent then the angles opposite the sides are congruent,

From these two theorems it follows that

<u>Theorem 3-5.</u> Two sides of a triangle are congruent if and only if the angles opposite them are congruent.

Using the definition of isosceles triangle, we see from Theorem 3-5 that a triangle is isosceles if and only if two of its angles are congruent. Contrast this with the definition.

Two congruent sides of an isosceles triangle are sometimes called <u>the legs of the isosceles triangle</u>. The third side is then called <u>the base</u>. The angles opposite the legs are called <u>the base angles</u>, and the angle opposite the base is called <u>the vertex angle</u>.

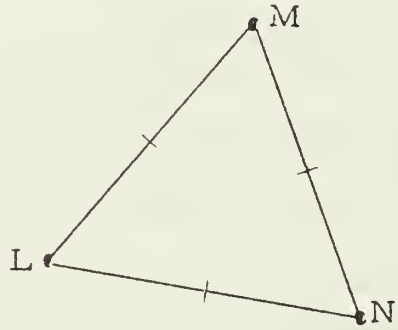


So, one of the theorems you proved [which one?] in Parts B and C on page 6-102 can be stated as follows:

The base angles of an isosceles triangle are congruent.

Which part of Theorem 3-5 [its if-part or its only-if-part] tells you this? What tells you that the legs of an isosceles triangle are congruent?

Here is a picture of $\triangle LMN$. Its three sides are congruent.



According to the definition, $\triangle LMN$ is isosceles. If we choose to say that $\overline{LM}$ and $\overline{LN}$ are its legs then what are its base angles? If we say that $\overline{MN}$ and $\overline{ML}$ are its legs, what is its vertex angle?

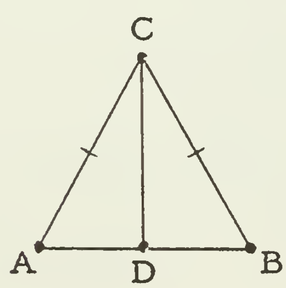
A triangle which, like $\triangle LMN$, has three congruent sides is called <u>an equilateral triangle</u>. According to the definitions, is an equilateral triangle an isosceles triangle?

Prove the following <u>corollary</u> of Theorem 3-5. [Guess what 'equiangular' means?]

<u>Theorem 3-6.</u> A triangle is equilateral if and only if it is equiangular.

*

<u>Example.</u>



<u>Hypothesis:</u> $\triangle ABC$ is isosceles, $\angle C$ is its vertex angle, D is the midpoint of $\overline{AB}$

<u>Conclusion:</u> $\angle DCA \cong \angle DCB$

Plan. Try to show that $\angle DCA$ and $\angle DCB$ are corresponding parts with respect to some congruence for ΔDCA and ΔDCB. The possible congruences are $DCA \longleftrightarrow DCB$ and $DCA \longleftrightarrow BCD$. The figure suggests that it must be the first. We know that $\overline{CA} \cong \overline{CB}$ and that $\angle A \cong \angle B$. So, the triangles agree in a pair of sides and a pair of angles. So, try either s.a.s. or a.s.a. Since we are also given that $\overline{AD} \cong \overline{BD}$, let's use s.a.s.

Solution I.

(1)	$\overline{CA} \cong \overline{CB}$	[Hypothesis; def. of isos. triangle]
(2)	Two sides of a triangle are congruent if and only if the angles opposite them are congruent.	[theorem]
(3)	$\angle A \cong \angle B$	[(1) and the only-if-part of (2)]
(4)	$\overline{AD} \cong \overline{BD}$	[Hypothesis; def. of midpoint]
(5)	C, A, D and C, B, D are vertices of triangles	[figure]
(6)	s.a.s.	[theorem]
(7)	$CAD \longleftrightarrow CBD$ is a congruence	[(5), (1), (3), (4), and (6)]
(8)	$\angle DCA \cong \angle DCB$	[(7); def. of congruence]

Solution II.

By hypothesis, ΔACB is isosceles with $\overline{CA} \cong \overline{CB}$. So, $\angle A \cong \angle B$ (1). Also, by hypothesis, $\overline{AD} \cong \overline{BD}$. Hence, by s.a.s., $CAD \longleftrightarrow CBD$ is a congruence. Therefore, $\angle DCA \longleftrightarrow \angle DCB$.

---

(1) Two sides of a triangle are congruent if and only if the angles opposite them are congruent. [Or: The base angles of an isosceles triangle are congruent.]

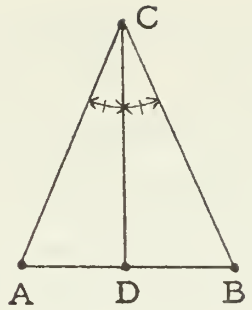
EXERCISES

A. 1. If ΔABC is equilateral, how many matchings among its vertices are congruences?

2. If ΔABC is isosceles, how many matchings among its vertices are congruences?

3. If ΔABC is scalene, how many matchings among its vertices are congruences?

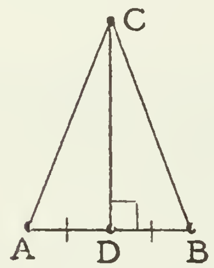
B. 1.



Hypothesis: $\overline{CD} \perp \overline{AB}$ at D, $\angle ACD \cong \angle BCD$

Conclusion: $\triangle ACB$ is isosceles

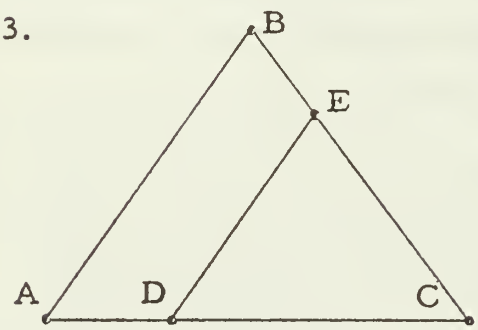
2.



Hypothesis: $\overline{CD}$ is the perpendicular bisector of $\overline{AB}$

Conclusion: $\triangle ACB$ is isosceles

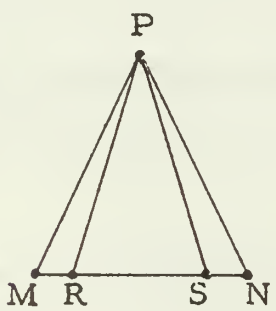
3.



Hypothesis: $\overline{AB} \cong \overline{BC}$, $\overline{DE} \cong \overline{EC}$

Conclusion: $\angle A \cong \angle EDC$

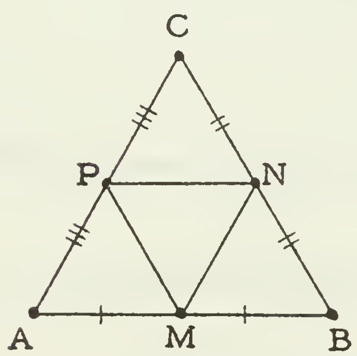
4.



Hypothesis: $\angle PRN \cong \angle PSM$, $\angle MPR \cong \angle NPS$

Conclusion: $\triangle MPN$ is isosceles

☆5.

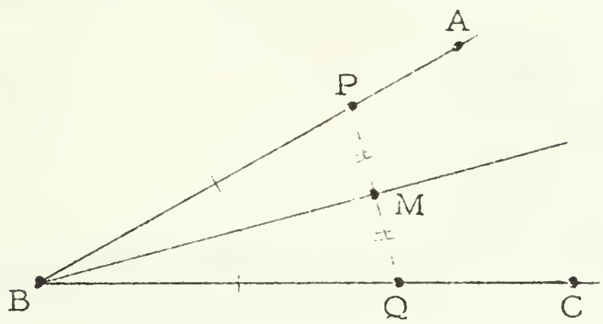


Hypothesis: $\triangle ABC$ is equilateral, M, N, and P are the midpoints of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively

Conclusion: $\triangle NPM$ is equilateral

ANGLE BISECTORS

As we showed in the Example on page 6-104,



if P and Q are points of $\overrightarrow{BA}$ and $\overrightarrow{BC}$ which are equidistant from B, and M is the midpoint of $\overline{PQ}$, then $\angle ABM \cong \angle CBM$. From the definition of interior of an angle, $\overrightarrow{BM}$ is contained in the interior of $\angle ABC$. Because $\angle ABM \cong \angle CBM$ and because $\overrightarrow{BM}$ is in the interior of $\angle ABC$, we say that $\overrightarrow{BM}$ is a *bisector* of $\angle ABC$.

Is there another ray, $\overrightarrow{BX}$, which is a bisector of $\angle ABC$? The answer to this question is 'no'. Here's why.

By the definition of a bisector of an angle, it follows from Axiom F that if $\overrightarrow{BX}$ is a bisector of $\angle ABC$ then $m(\angle ABX) = \frac{1}{2} \cdot m(\angle ABC)$. By the definition of interior of an angle, it follows from Axioms D and E that there is at most one half-line h in the interior of $\angle ABC$ such that $m(\overrightarrow{BA} \cup h) = \frac{1}{2} \cdot m(\angle ABC)$. So, an angle has at most one angle bisector.

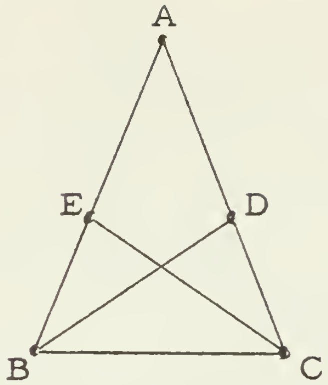
As in the case of the midpoint of a segment, we can now speak of *the* bisector of an angle. So, we have the following theorem:

> *Theorem 3-7.* [Definition of angle bisector]
>
> For each angle $\angle XYZ$, and for each half-line h with vertex Y, the ray $h \cup \{Y\}$ is the bisector of $\angle XYZ$ if and only if h is in the interior of $\angle XYZ$ and $h \cup \overrightarrow{YX} \cong h \cup \overrightarrow{YZ}$.

We have also proved:

> *Theorem 3-8.*
>
> The bisector of an angle and either side of the angle are the sides of an angle whose measure is half that of the given angle.

Example.



Hypothesis: $\triangle BAC$ is an isosceles triangle with $\angle A$ as vertex angle,
$\overrightarrow{BD}$ and $\overrightarrow{CE}$ are the bisectors of the base angles

Conclusion: $BD = CE$

Plan. Find a matching for which $\overline{BD}$ and $\overline{CE}$ are corresponding parts, and show that the matching is a congruence. There are several possibilities. Two of them are $BDA \longleftrightarrow CEA$ and $BDA \longleftrightarrow ECA$. The figure suggests the first of these. $\overline{BA}$ and $\angle A$ of $\triangle BDA$ are congruent to $\overline{CA}$ and $\angle A$, respectively, of $\triangle CEA$. Should we use s.a.s.? Then, we would have to show that $\overline{DA} \cong \overline{EA}$. That looks hard. Could we use a.s.a.? We could if we could show that $\angle ABD \cong \angle ACE$. This is easy because $\angle ABC \cong \angle ACB$ and $\overrightarrow{BD}$ and $\overrightarrow{CE}$ are their bisectors.

Solution I.

(1)	$\angle A \cong \angle A$	[Identity; def. of congruent angles]
(2)	$\overline{BA} \cong \overline{CA}$	[Hypothesis; def. of isosceles triangle]
(3)	Two sides of a triangle are congruent if and only if the angles opposite them are congruent.	[theorem]
(4)	$\angle ACB \cong \angle ABC$	[(2) and the only-if-part of (3)]
(5)	$\overrightarrow{CE}$ is the bisector of $\angle ACB$	[Hypothesis]
(6)	The bisector of an angle and either side of the angle are the sides of an angle whose measure is half that of the given angle.	[theorem]
(7)	$m(\angle ACE) = \frac{1}{2} \cdot m(\angle ACB)$	[(5) and (6)]
(8)	$m(\angle ABD) = \frac{1}{2} \cdot m(\angle ABC)$	[steps like (5) and (6)]
(9)	$\angle ABD \cong \angle ACE$	[(4), (7) and (8); def. of congruent angles]

(10) B, D, A and C, E, A are vertices of triangles	[figure]
(11) a.s.a.	[theorem]
(12) BDA $\longleftrightarrow$ CEA is a congruence	[(10), (9), (2), (1), and (11)]
(13) BD = CE	[(12); def. of congruence; def. of congruent segments]

<u>Solution II.</u>

Since $\angle ABC$ and $\angle ACB$ are base angles of an isosceles triangle, they are congruent (1). Since $\overrightarrow{BD}$ and $\overrightarrow{CE}$ are the bisectors of these angles, $\angle ABD$ and $\angle ACE$ are congruent, also (2). Consequently, in $\triangle BDA$ and $\triangle CEA$, $\angle B \cong \angle C$, $\overline{BA} \cong \overline{CA}$, and $\angle A \cong \angle A$. Hence, by a.s.a. BDA $\longleftrightarrow$ CEA is a congruence. So, BD = CE.

---

(1) The base angles of an isosceles triangle are congruent.

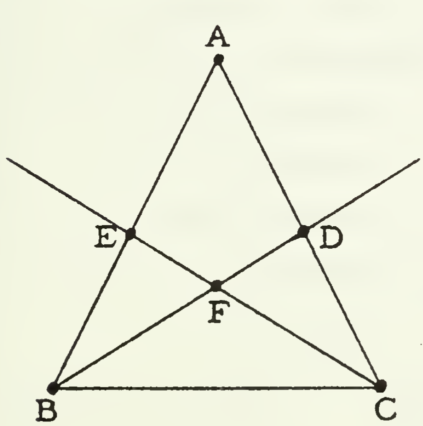
(2) The bisector of an angle and either side of the angle are the sides of an angle whose measure is half that of the given angle.

EXERCISES

<u>A.</u> The diagram on page 6-107 suggests a way of using compass and straight-edge alone to draw the bisector of an angle, $\angle ABC$. Use the compass to locate points P and Q such that $\overline{BP} \cong \overline{BQ}$. So, B is one of the points on the perpendicular bisector of $\overline{PQ}$. Use the compass to locate a second point on the perpendicular bisector of $\overline{PQ}$. This second point and B determine the angle bisector. Why?

Use this technique to bisect acute, right, and obtuse angles.

<u>B.</u> 1.



<u>Hypothesis:</u> $\triangle ABC$ is isosceles with $\overline{AB} \cong \overline{AC}$, $\overrightarrow{BD}$ and $\overrightarrow{CE}$ are the bisectors of the base angles and intersect at F

<u>Conclusion:</u> $\triangle BFC$ is isosceles

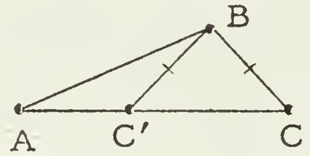
2. In Exercise 1, show that $\overrightarrow{AF}$ is the bisector of $\angle A$.

C. 1. $\angle ABC$ and $\angle CBD$ are adjacent supplementary angles. Show that their angle bisectors are perpendicular. [Hint. Draw a diagram, label the bisectors, and state the hypothesis and conclusion.]

2. $\angle ABC$ and $\angle DBE$ are vertical angles. Show that their angle bisectors are collinear.

3. Lines ℓ and m intersect in a point P. Show that the bisectors of the four angles contained in $\ell \cup m$ form two perpendicular lines. [Hint. You may refer to the results obtained in Exercises 1 and 2.]

D.



Hypothesis: $BC' = BC$, $C' \in \overline{AC}$

Conclusion: $\angle AC'B$ and $\angle C$ are supplementary

SUMMARY OF SECTION 6.03

Notation and terminology

angle bisector	[6-107]	isosceles triangle	[6-103]
bisector	[6-90]	base of	[6-103]
of an angle	[6-107]	base angles of	[6-103]
of a segment	[6-90]	legs of	[6-103]
perpendicular	[6-94]	vertex angle of	[6-103]
congruence	[6-82]	opposite angle	[6-79]
congruent triangles	[6-82]	opposite side	[6-79]
corresponding parts	[6-81]	perpendicular bisector	[6-94]
equiangular triangle	[6-104]	scalene triangle	[6-103]
equilateral triangle	[6-104]	triangle	[6-79]
exterior of a triangle	[6-93]	angle of	[6-79]
included angle	[6-87]	exterior of	[6-93]
included side	[6-87]	interior of	[6-93]
interior of a triangle	[6-93]	side of	[6-79]
matching of vertices	[6-80]	vertex of	[6-79]

$\triangle ABC$	[6-79]	$ABC \longleftrightarrow A'B'C'$	[6-80]	$\triangle ABC \cong \triangle A'B'C'$	[6-83]
a.s.a.	[6-98]	s.a.s.	[6-86]	s.s.s.	[6-86]

Theorems

3-1. If, for some matching of the vertices of one triangle with those of a second, each side of the first triangle is congruent to the corresponding side of the second, then this matching is a congruence. [s.s.s.]

3-2. If, for some matching of the vertices of one triangle with those of a second, each of two sides of one triangle is congruent to the corresponding side of the second, and the included angles are congruent, then this matching is a congruence. [s.a.s.]

3-3. A point is equidistant from the two end points of a segment if and only if it belongs to the perpendicular bisector of the segment.

3-4. If, for some matching of the vertices of one triangle with those of a second, each of two angles of one triangle is congruent to the corresponding angle of the second, and the included sides are congruent, then this matching is a congruence. [a.s.a.]

3-5. Two sides of a triangle are congruent if and only if the angles opposite them are congruent.

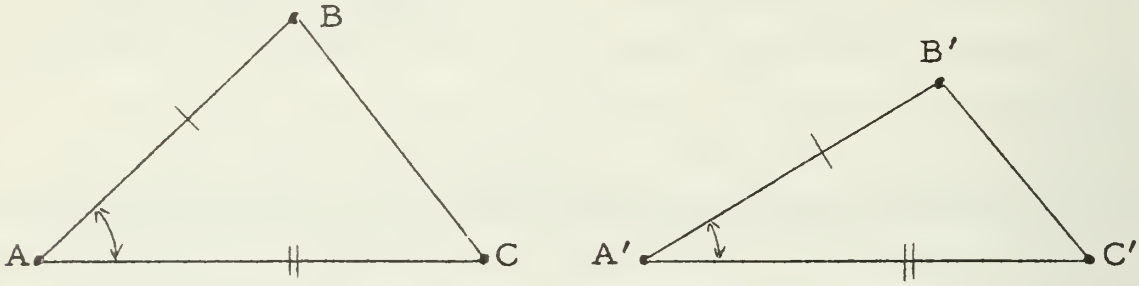
3-6. A triangle is equilateral if and only if it is equiangular.

3-7. For each angle $\angle XYZ$, and for each half-line h with vertex Y, the ray $h \cup \{Y\}$ is the bisector of $\angle XYZ$ if and only if h is in the interior of $\angle XYZ$ and $h \cup \overrightarrow{YX} \cong h \cup \overrightarrow{YZ}$.

3-8. The bisector of an angle and either side of the angle are the sides of an angle whose measure is half that of the given angle.

[Supplementary exercises are on page 6-419.]

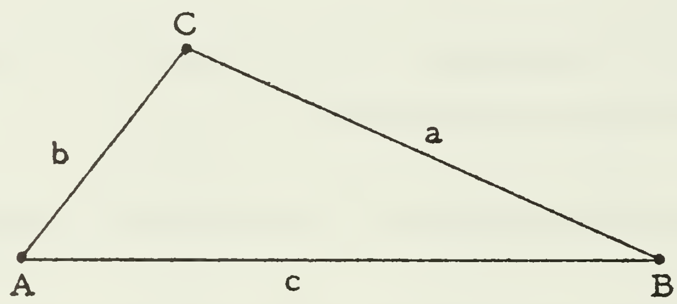
6.04 <u>Geometric inequations.</u> --The work on congruence has dealt with equality of measures of segments and of angles. It is also important to consider inequality of measures. For example, suppose two triangles agree in two pairs of corresponding sides but the measure of



the included angle of one is greater than the measure of the included angle of the other. What conclusion do you think follows about the measures of the third sides? Later in this section we shall prove a theorem dealing with this problem.

AN INEQUATION FOR THE SIDES OF A TRIANGLE

One of the more obvious inequations for measures of geometric figures is suggested by Axiom B. Given a triangle with vertices A, B, and C and side-measures a, b, and c as shown.



According to Axiom B, since $C \notin \overleftrightarrow{AB}$, it follows that $b + a > c$. So, the sum of the measures of two sides of a triangle is greater than the measure of the third. This last statement is often abbreviated to:

> <u>Theorem 4-1.</u>
> The sum of two sides of a triangle is greater than the third.

[Of course, the word 'sum' refers to measures even though, for the sake of brevity, the word 'measure' doesn't occur in the theorem.]

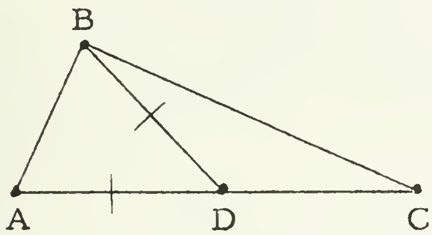
EXERCISES

A. For each ordered triple listed below, tell whether its components can be measures of the sides of a triangle.

1. (2, 5, 8)	2. (9, 11, 20)	3. (8, 8, 16)
4. (16, 16, 8)	5. (3, 3, 3)	6. (3, 4, 5)
7. (1, 2, 3)	8. (10, 20, 29)	9. (6, 12, 4)

B. 1. Show that the difference of two sides of a triangle is less than the third side.

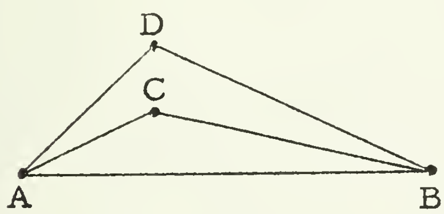
2.



Hypothesis: AD = BD

Conclusion: BC < AC

☆ 3.

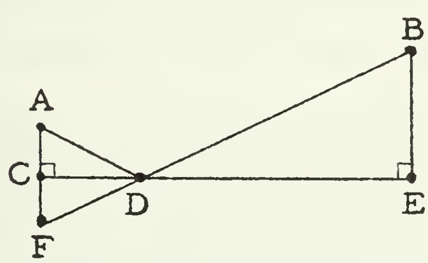


Hypothesis: C is a point in the interior of $\triangle ABD$

Conclusion: AC + CB < AD + DB

[Hint. Since C is in the interior of $\triangle ABD$, the half-line $\overrightarrow{AC}$ inter- intersects $\overleftrightarrow{BD}$. Let E be the point of intersection...]

☆ C. 1.

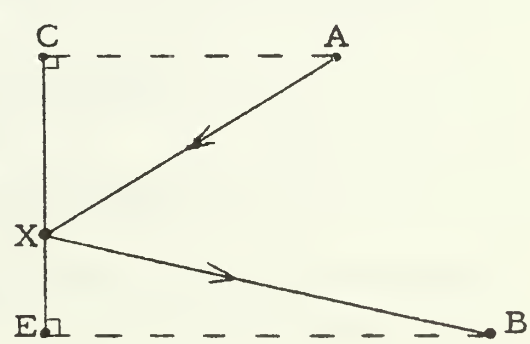


Hypothesis: $\overleftrightarrow{AC} \perp \overleftrightarrow{CE}$, C is the midpoint of $\overleftrightarrow{AF}$, F, D, and B are collinear

Conclusion: FB = AD + DB

2. In Exercise 1, suppose P is a point of $\overleftrightarrow{CE}$ other than D. Show that AP + PB > AD + DB.

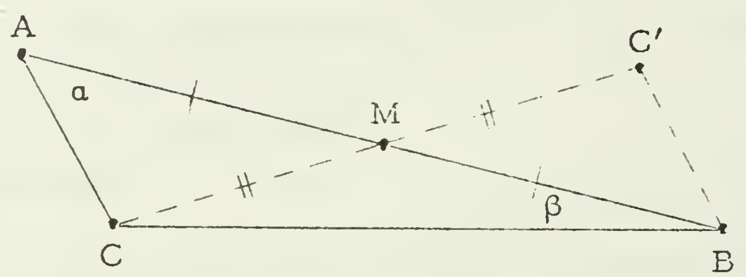
3. For each point $X \in \overleftrightarrow{CE}$, a "trip" from A to X and from X to B is AX + XB units long. Locate that point of $\overleftrightarrow{CE}$ which gives the shortest trip. Then, show that if Q is such a point then $\angle AQC \cong \angle BQE$



INEQUATIONS FOR THE ANGLES OF A TRIANGLE

Theorem 4-1 tells us that the sum of each two sides of a triangle is greater than the third side. Do you think there is a similar theorem about the angles of a triangle? Is the sum of the measures of two angles of a triangle always greater than the measure of the third angle? Can you draw a triangle for which this is not the case?

There is an important theorem concerning the sum of the measures of two angles of a triangle. Consider $\triangle ABC$ where $m(\angle A) = \alpha$ and $m(\angle B) = \beta$. [Just as Roman letters like 'a', 'b', and 'c', are used as variables for measures of segments, Greek letters like 'α' ["alpha"], 'β' ["beta"], and 'γ' ["gamma"] are sometimes used as variables for measures of angles.]



Let M be the midpoint of $\overline{AB}$ and let C' be the point on $\overrightarrow{CM}$ such that $MC' = MC$.

So, for the triangles, $\triangle CAM$ and $\triangle C'BM$, $CAM \longleftrightarrow C'BM$ is a congruence. [Why?] From this it follows that $\angle CAM \cong \angle C'BM$. That is, $m(\angle C'BM) = \alpha$.

Now, since $B \notin \overleftrightarrow{CC'}$ [Why?], $\overrightarrow{BC} \cup \overrightarrow{BC'}$ is an angle. Since M is interior to $\angle C'BC$, $m(\angle C'BM) + m(\angle MBC) = m(\angle C'BC)$. That is, $\alpha + \beta = m(\angle C'BC)$. But, by Axiom D, $m(\angle C'BC) < 180$. So,

$$\alpha + \beta < 180.$$

Thus, we have proved the following theorem:

Theorem 4-2. The sum of the measures of two angles of a triangle is less than 180.

This theorem has several important corollaries which you will prove in the following exercises.

EXERCISES

A. Prove:

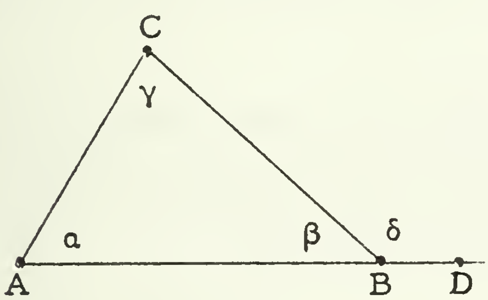
Theorem 4-3.
No two angles of a triangle are supplementary.

B. Prove:

Theorem 4-4.
If one angle of a triangle is a right angle or an obtuse angle then the others are acute angles.

C. Prove that the sum of the measures of the angles of a triangle is less than 270.

D.



Hypothesis: $B \in \overline{AD}$, $m(\angle A) = \alpha$, $m(\angle C) = \gamma$, $m(\angle ABC) = \beta$, $m(\angle CBD) = \delta$

Conclusion: $\delta > \alpha$, $\delta > \gamma$
['δ' is read as 'delta']

[Hint. Since $\delta + \beta = 180$ and $\alpha + \beta < 180$, it follows that ...]

* * *

In Part D, the angle with sides $\overrightarrow{BD}$ and $\overrightarrow{BC}$ is an exterior angle of ΔABC. An angle is an exterior angle of a triangle if and only if it is adjacent and supplementary to one of the angles of the triangle. The other two angles of the triangle are said to be opposite the exterior angle. So, what you have proved in Part D is that the measure of an exterior angle of a triangle is greater than the measure of either of the opposite angles of the triangle.

Theorem 4-5.
An exterior angle of a triangle is larger than each of the angles opposite it.

* * *

E. 1. Draw a triangle one of whose exterior angles is an angle of 150°.

2. Repeat for an angle of 20°. 3. Repeat for an angle of 90°.

4. If an exterior angle of a triangle is acute, how many acute angles does the triangle have? How many obtuse angles?

5. If an exterior angle of a triangle is obtuse, how many acute angles does it have? Can it have a right angle? Can it have an obtuse angle?

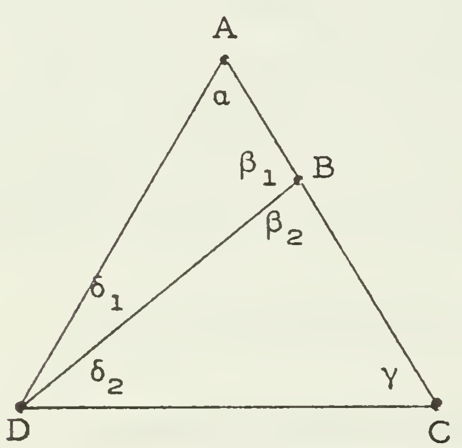
6. An acute triangle is one for which all three angles are acute. Draw an acute triangle. Tell why all of the exterior angles of an acute triangle are obtuse.

7. Tell why a triangle with a right angle must have obtuse exterior angles at two of its vertices.

8. An obtuse triangle is one for which one angle is obtuse. Draw an obtuse triangle. Tell why an obtuse triangle must have obtuse exterior angles at two of its vertices.

F. If $\triangle ABC$ has two congruent exterior angles, does it follow that $\triangle ABC$ is isosceles? If not, what additional information do you need to have about the congruent exterior angles in order to conclude that $\triangle ABC$ is isosceles?

G.



Hypothesis: $\overleftrightarrow{AD} \cong \overleftrightarrow{AC}$, $B \in \overline{AC}$

Conclusion: $\beta_1 > \delta_1$

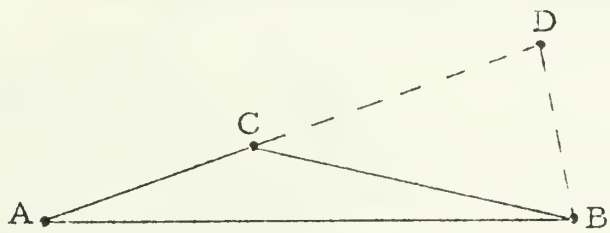
[Hint. Show that $\beta_1 > \gamma > \delta_1$.]

☆H. Refer to the diagram on page 6-114, and show that $CM < \frac{1}{2}(CA + CB)$.

COMPARING SIDES OR ANGLES OF A TRIANGLE

You have proved that if two sides of a triangle are congruent then so are the angles opposite the sides. But, what can you say about the angles opposite two sides which are not congruent? Clearly, the angles are not congruent. [Why?] Draw some pictures and see if you can say more than this.

Consider $\triangle ABC$ for which $\overline{AB}$ is longer than $\overline{AC}$. It looks as if



$\angle ACB$ is larger than $\angle ABC$. Let's prove this.

Let D be the point of $\overrightarrow{AC}$ such that $AD = AB$. Since $C \in \overline{AD}$, $\angle ACB$ is larger than $\angle D$ [Why?]. Since $\angle D \cong \angle ABD$ [Why?], $\angle ACB$ is larger than $\angle ABD$. Since C is interior to $\angle ABD$, Axiom F tells us that $\angle ABD$ is larger than $\angle ABC$. Hence, $\angle ACB$ is larger than $\angle ABC$.

So, we have proved the following theorem:

> <u>Theorem 4-6.</u>
>
> If one side of a triangle is longer than another then the angle opposite the first is larger than the angle opposite the second.

Now, let's consider the converse. Suppose, in $\triangle ABC$, $\angle C$ is larger than $\angle B$. Then, $\angle C \not\cong \angle B$. So, $\overline{AB} \not\cong \overline{AC}$. [Why?] Also, $\angle B$ is not larger than $\angle C$. So, $\overline{AC}$ is not longer than $\overline{AB}$. [Why?]

Since $\overline{AB} \not\cong \overline{AC}$ and $\overline{AC}$ is not longer than $\overline{AB}$, it follows that $\overline{AB}$ is longer than $\overline{AC}$. [Why?] So, we have proved:

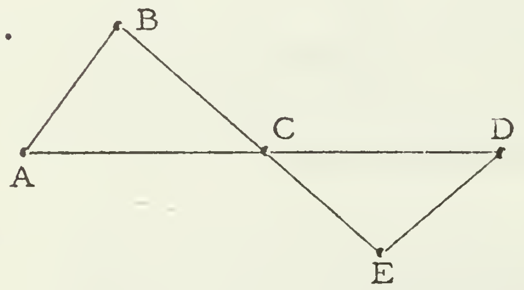
> <u>Theorem 4-7.</u>
>
> If one angle of a triangle is larger than another then the side opposite the first is longer than the side opposite the second.

EXERCISES

A. 1. In ΔMRT, $MR = 6$, $RT = 5$, and $TM = 8$. What is the largest angle of this triangle? The smallest?

2. In ΔTUB, $m(\angle T) = 100$, $m(\angle U) = 20$, and $m(\angle B) = 60$. What is the longest side of this triangle? The shortest?

3. If, in ΔCAR, $m(\angle C) \leq m(\angle A)$, what can you say about the measures of the sides opposite $\angle C$ and $\angle A$?

4. 

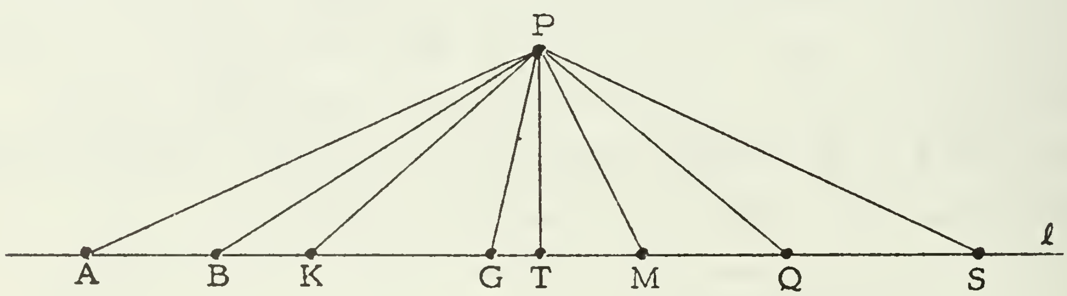
Suppose that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{CD}$, $\overline{BC}$ is longer than $\overline{DE}$, $\overline{AC}$ is longer than $\overline{CD}$, and $\overline{AB}$ is longer than $\overline{CE}$. Which of the six angles of ΔABC and ΔCDE is the smallest?

B. 1. Suppose that ΔABC is an isosceles triangle with vertex angle at A, and that D is a point on $\overrightarrow{BC}$ such that $C \in \overline{BD}$. Show that $AD > AB$.

2. Suppose that D is a point in the interior of $\angle BAC$ and that A is a point in the interior of $\angle BDC$. Suppose, further, that $AB = AC$, $AC < CD$, and $CD = BD$. Which is larger, $\angle A$ or $\angle D$? Prove it.

THE PERPENDICULAR THROUGH A POINT

Suppose that ℓ is a line and P is a point not on ℓ. Then, for each point on ℓ, there is a segment with P and this point as end points.



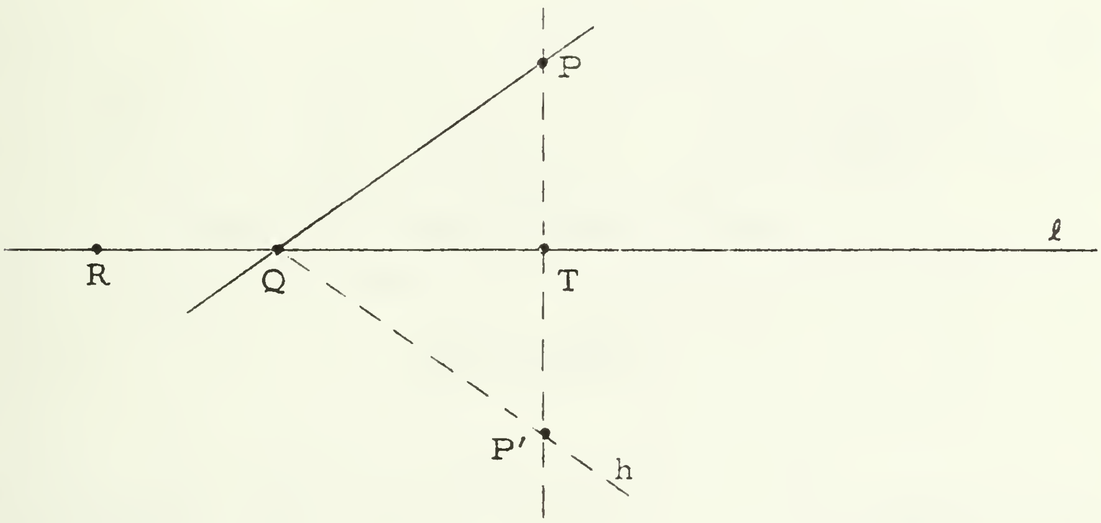
Do you think one of the segments is shorter than each of the others?

You have probably guessed that there is a shortest segment, and, in fact, that it is the one which is perpendicular to ℓ.

It is easy to see that <u>if</u> $\overline{PT}$ is perpendicular to ℓ then any other segment is longer. For example, take $\overline{PM}$. In ΔPTM, $\angle T$ is a right angle. So, by Theorem 4-4, $\angle PMT$ is an acute angle. Hence, $\angle T$ is larger than $\angle PMT$, and by Theorem 4-7, $\overline{PM}$ is longer than $\overline{PT}$.

So, we have shown that <u>if</u> there is a point $T \in \ell$ such that $\overleftrightarrow{PT} \perp \ell$ then $\overline{PT}$ is the shortest segment from P to ℓ. But, is there such a point T? In other words, given a line ℓ and a point $P \notin \ell$, is there a line m such that $P \in m$ and $m \perp \ell$? Notice that by the result obtained in the preceding paragraph, there can't be two lines through P perpendicular to ℓ. [Explain.]

Let's prove that there is a perpendicular from P to ℓ. Suppose R and Q are two points of ℓ. Either $\overleftrightarrow{PQ} \perp \ell$ or $\overleftrightarrow{PQ} \not\perp \ell$. If $\overleftrightarrow{PQ} \perp \ell$ then there certainly is a line through P perpendicular to ℓ. So, suppose $\overleftrightarrow{PQ} \not\perp \ell$.



By Axioms D and E, there is one and only one half-line h with vertex Q which is contained in the non-P-side of ℓ and which is such that $h \cup \overrightarrow{QR} \cong \angle PQR$. Let P′ be the point of h such that $\overline{QP'} \cong \overline{QP}$ [Axiom C]. Now, $\overline{PP'}$ intersects ℓ [Why?] at some point T. Since $\angle PQT \cong \angle P'QT$ [Why?], it follows by s.a.s. that $PQT \longleftrightarrow P'QT$ is a congruence. Hence, $\angle PTQ \cong \angle P'TQ$. So, both these angles are right angles [Why?], and $\overleftrightarrow{PT} \perp \ell$. [The point T is called <u>the foot of the perpendicular</u>.]

The measure of the shortest segment from a point to a line is called <u>the distance between the point and the line</u>. For the figure above, the distance between P and ℓ is PT. What is the distance between Q and $\overleftrightarrow{PP'}$? What is the distance between R and $\overleftrightarrow{PP'}$?

So, for each line ℓ and each point $P \notin \ell$, there is one and only one line through P which is perpendicular to ℓ. As we have shown earlier, this is also the case if $P \in \ell$. Hence, for each line ℓ and each point P, we are entitled to speak of <u>the</u> perpendicular to ℓ through P. So, we have the following theorem:

<u>Theorem 4-8.</u> [Definition of perpendicular through a point]

$\forall_m \forall_\ell \forall_P$

m is the perpendicular to ℓ through P
if and only if
$P \in m$ and $m \cup \ell$ contains a right angle

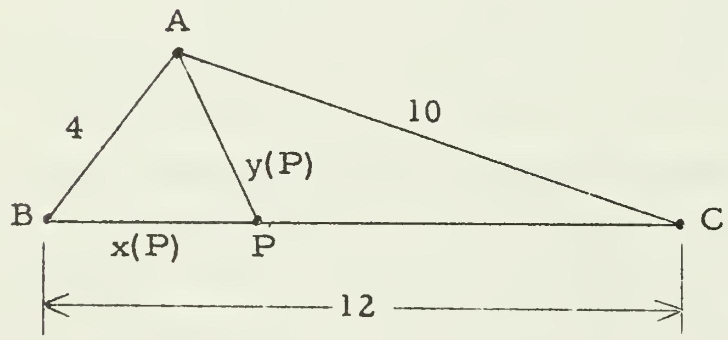
We have also proved the following:

<u>Theorem 4-9.</u>

The shortest segment from a point to a line is the one which is perpendicular to the line.

EXERCISES

<u>A.</u>



For each point $P \in \overleftrightarrow{BC}$, let $x(P) = BP$ and $y(P) = AP$.

1. (a) If P is the midpoint of $\overleftrightarrow{BC}$ then $x(P) =$ ______.

 (b) If P_1 and P_2 are points of $\overleftrightarrow{BC}$ such that $BP_1 = P_1P_2 = P_2C$ then $x(P_1) =$ ______ and $x(P_2) =$ ______.

 (c) $x(B) =$ ______ and $y(B) =$ ______.

 (d) $x(C) =$ ______ and $y(C) =$ ______.

2. (a) Draw horizontal and vertical axes on a sheet of cross-section paper with the origin near the lower left corner of the paper. Copy ΔABC with B at the origin and C on the horizontal axis. Now, graph the set of ordered pairs $(x(P), y(P))$. [You can plot lots of these ordered pairs with just one swing of your compass for each.]

   (b) There should be a lowest point on your graph. Why?

3. Now, suppose that T is the foot of the perpendicular from A to $\overleftrightarrow{BC}$.

   (a) If $BT = TP$ then $y(P) =$ ________.

   (b) If P_1 and P_2 are points of $\overline{BC}$ such that T is the midpoint of $\overline{P_1P_2}$, what do you think is the relation between $y(P_1)$ and $y(P_2)$? Justify your guess.

   (c) If P_1 and P_2 are points of $\overline{BC}$ such that $\overline{P_2T}$ is longer than $\overline{P_1T}$, what do you think is the relation between $y(P_2)$ and $y(P_1)$? [Choose $P_3 \in \overline{TP_2}$ such that $P_1T = TP_3$. Then, use Theorems 4-5 and 4-7 to show that $y(P_2) > y(P_3)$.]

4. Now, suppose that $\angle APB \cong \angle B$. Then, $y(P) =$ ______.

5. Notice that $\angle B$ is larger than $\angle C$. [What tells you this?]

   (a) If $P \in \overline{BC}$, is $\angle APC$ larger than $\angle C$? Why?

   (b) If $P \in \overline{BC}$, does it follow that $\overline{AC}$ is longer than $\overline{AP}$? Why?

6. If, in ΔABC, side $\overline{AC}$ had been congruent to side $\overline{AB}$, would it still follow that if $P \in \overline{BC}$ then $\overline{AC}$ is longer than $\overline{AP}$? Justify your answer.

* * *

Your work in Exercises 5 and 6 gives us the following theorem:

<u>Theorem 4-10.</u>

For each ΔXYZ and each $W \in \overline{YZ}$, if $\overline{XY}$ is not longer than $\overline{XZ}$ then $\overline{XZ}$ is longer than $\overline{XW}$.

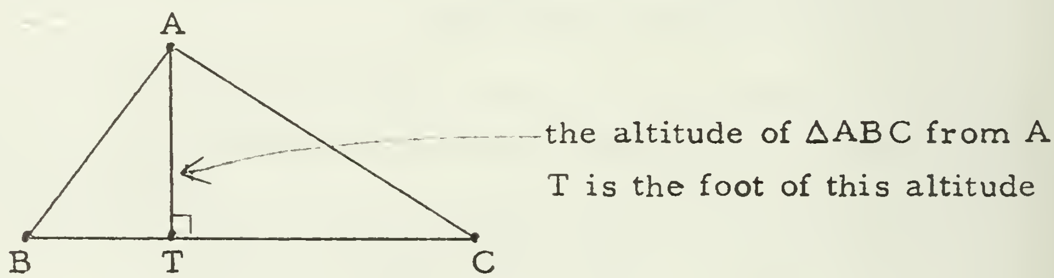
* * *

B. 1. Draw a line ℓ and a point P not on ℓ. Use compass and straight-edge to draw the perpendicular through P to ℓ.

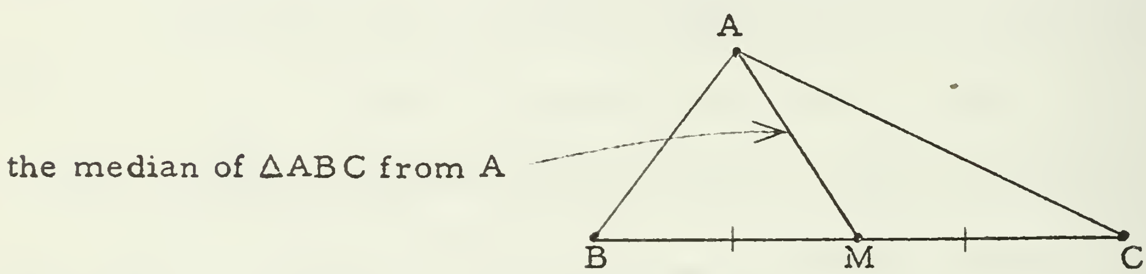
2. Draw an acute triangle $\triangle ABC$. Use compass and straight-edge to draw the segment from B which is perpendicular to $\overleftrightarrow{AC}$. Then, draw the segment from C perpendicular to $\overleftrightarrow{AB}$. Finally, draw the shortest segment from A to $\overleftrightarrow{BC}$.

ALTITUDE, MEDIAN, ANGLE BISECTOR

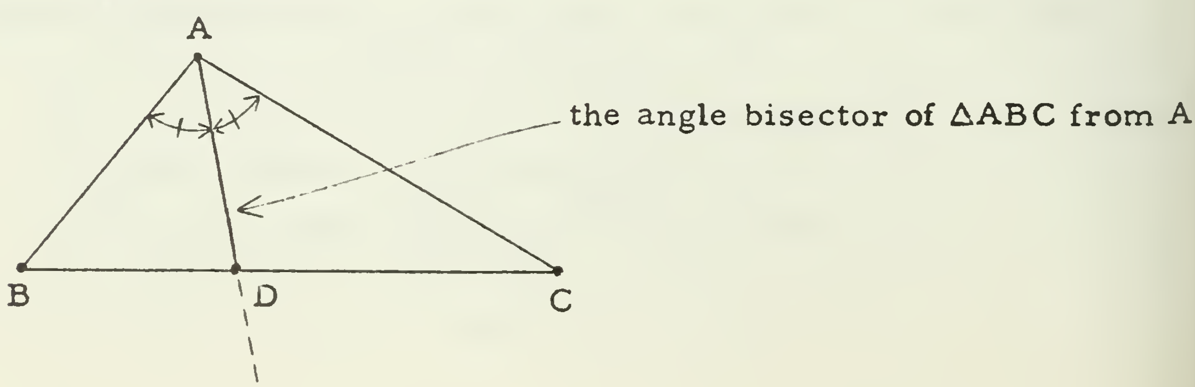
The segment from a vertex of a triangle perpendicular to the line containing the opposite side is called <u>the altitude of the triangle from the given vertex</u>.



The segment from a vertex of a triangle to the midpoint of the opposite side is called <u>the median of the triangle from the given vertex</u>.



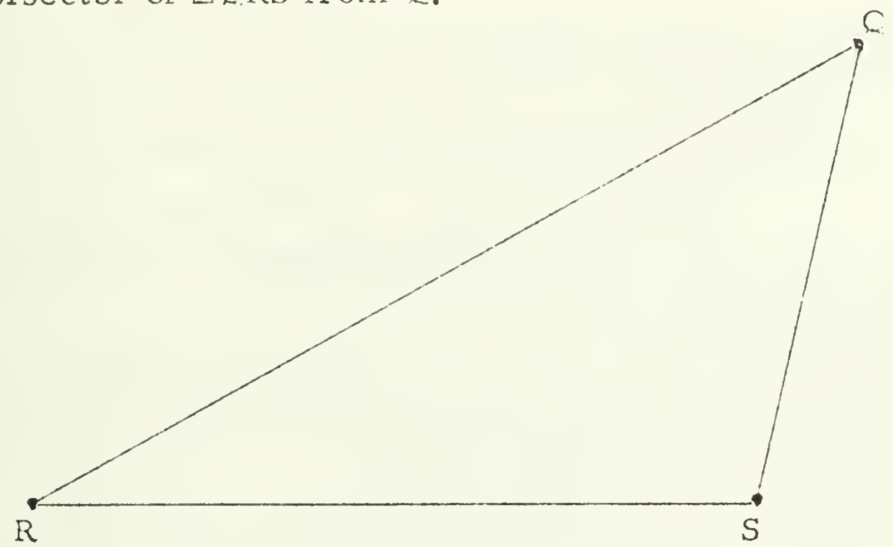
The segment from a vertex of a triangle, along the bisector of the angle, to the opposite side is called <u>the angle bisector of the triangle from the given vertex</u>.



[That the angle bisector of an angle of a triangle intersects the opposite side follows from the definition of the bisector of an angle and Introduction Axioms.]

EXERCISES

A. 1. Copy $\triangle QRS$, and then draw the median, the altitude, and the angle bisector of $\triangle QRS$ from Q.



2. Repeat Exercise 1 for a $\triangle QRS$ where $\angle QSR$ is a right angle.

3. Repeat Exercise 1 for an isosceles triangle with vertex angle at Q.

B. 1. Prove that if $\triangle ABC$ is an isosceles triangle and $\angle A$ is its vertex angle then the angle bisector, the median, and the altitude of $\triangle ABC$ from A are the same segment. [Hint. It is enough to prove that the angle bisector is the altitude and that the median is the altitude.]

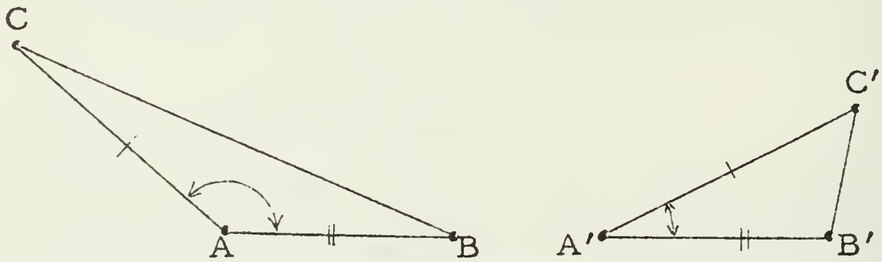
2. Prove that if the altitude of $\triangle ABC$ from A is the angle bisector of $\triangle ABC$ from A then $\triangle ABC$ is isosceles with vertex angle at A.

3. Prove that if the altitude of $\triangle ABC$ from A is the median of $\triangle ABC$ from A then $\triangle ABC$ is isosceles with vertex angle at A.

[Note. It is natural to ask whether a triangle is isosceles if one of its angle bisectors is a median. This is so, but is hard to justify now. (Find out why, by trying to do so.) Later we shall have a theorem which will make the job easy.]

☆ C. Your experience with scissors should convince you that the wider the angle of opening, the farther apart are the points. This suggests the theorem we alluded to at the beginning of our discussion of geometric inequations [page 6-112]. Your answers to the questions below will lead to the theorem.

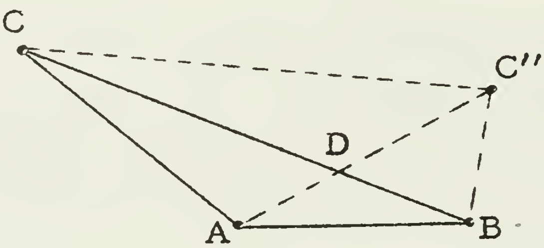
Suppose we have two triangles, ΔABC and $\Delta A'B'C'$, in which



$AC = A'C'$, $AB = A'C'$, and $\angle A$ is larger than $\angle A'$. Let's prove that $\overline{BC}$ is longer than $\overline{B'C'}$.

For definiteness, let's assume that $\overline{AB}$ is not longer than $\overline{AC}$. The same argument will apply in case $\overline{AC}$ is not longer than $\overline{AB}$. [Would we need to consider other cases?]

Rather than attempt to compare $\overline{BC}$ directly with $\overline{B'C'}$, it will probably be simpler to compare $\overline{BC}$ with a segment congruent to $\overline{B'C'}$ having B as one end point. This suggests considering a triangle $\Delta ABC''$ for which $ABC'' \longleftrightarrow A'B'C'$ is a congruence.



1. How do you know there is a point C'' in the C-side of $\overleftrightarrow{AB}$ such that $ABC'' \longleftrightarrow A'B'C'$ is a congruence?

2. How do you know that $C'' \notin \overrightarrow{AC}$?

3. How do you know that C is not in the interior of $\angle BAC''$? So, we conclude that $\angle BAC''$ and $\angle CAC''$ are adjacent angles.

4. $\overline{BC}$ crosses $\overrightarrow{AC''}$. Why?

5. Read Theorem 4-10 on page 6-121. Use it to show that $\overline{AD}$ is shorter than $\overline{AC''}$.

6. How do you know that D belongs to $\overline{AC''}$ as well as to $\overline{BC}$?

7. How do you know that A is in the interior of $\angle CC''B$ and that B is the interior of $\angle ACC''$?

8. $\angle ACC'' \cong \angle AC''C$. Why?

9. $\angle BCC''$ is smaller than $\angle ACC''$ and $\angle AC''C$ is smaller than $\angle BC''C$. Why?

10. Why does it now follow that $\overleftrightarrow{BC''}$ is shorter than $\overleftrightarrow{BC}$?

11. Why does it follow that $\overleftrightarrow{BC}$ is longer than $\overleftrightarrow{B'C'}$?

Thus, we have the following theorem:

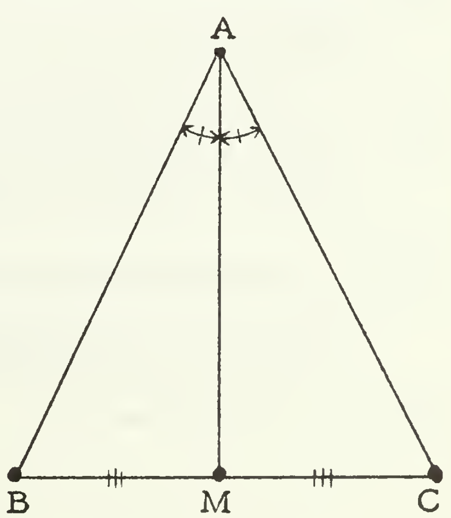
> <u>Theorem 4-11.</u>
>
> If two triangles agree in two pairs of sides but not in the included angles then the triangle with the larger included angle has the longer third side.

12. State and prove a theorem about two triangles which agree in two pairs of sides but not in the third pair of sides.

[<u>Hint.</u> Recall how Theorem 4-7 on page 6-117 was derived from Theorem 4-6.]

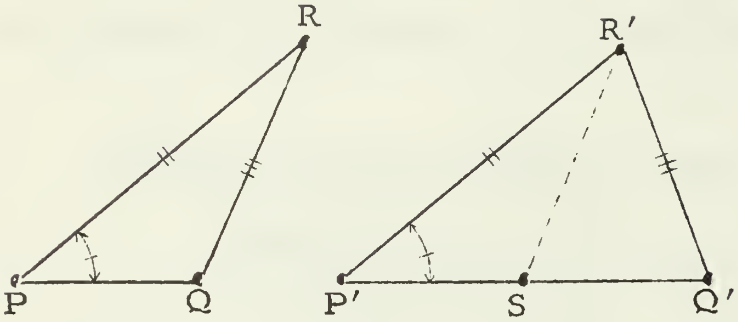
MORE TRIANGLE-CONGRUENCE THEOREMS

Suppose, in ΔABC, that $\overleftrightarrow{AM}$ is both the median and the angle bisector from A. Intuitively, it appears that $AMB \longleftrightarrow AMC$ is a congruence, and so, that ΔABC is isosceles. The triangles, ΔAMB and ΔAMC, do agree in the pairs of sides, $\overleftrightarrow{AM}$, $\overleftrightarrow{AM}$, and $\overleftrightarrow{MB}$, $\overleftrightarrow{MC}$. They also agree in a pair of angles. But, it is not known that they agree in the included angles, $\angle AMB$ and $\angle AMC$. So, the s.a.s. theorem does not apply. We need another theorem to handle this problem easily.



Consider two triangles, ΔPQR and $\Delta P'Q'R'$.

<u>Hypothesis</u>: $\angle P \cong \angle P'$
$PR = P'R'$
$RQ = R'Q'$



Conclusion: ______?______

This is a case of two triangles which agree in two pairs of sides and in the angles opposite the sides of one pair.

Suppose $\overleftrightarrow{PQ}$ is shorter than $\overleftrightarrow{P'Q'}$. Then, by Axiom C, there is a point S such that $S \in \overrightarrow{P'Q'}$ and $P'S = PQ$. By Theorem 1-5, it follows that $S \in \overline{P'Q'}$.

Now, in the triangles, ΔRPQ and $\Delta R'P'S$, we know, by hypothesis, that $RP = R'P'$ and $\angle P \cong \angle P'$. So, since $PQ = P'S$, it follows from s.a.s. that $RPQ \longleftrightarrow R'P'S$ is a congruence. Therefore, $RQ = R'S$.

Since, by hypothesis, $RQ = R'Q'$ also, it follows that $\angle R'SQ' \cong \angle Q'$. [Explain.]

Since $S \in \overline{P'Q'}$, the exterior angle $\angle P'SR'$ is supplementary to $\angle Q'$, also. But, since $RPQ \longleftrightarrow R'P'S$ is a congruence, $\angle Q \cong \angle P'SR'$. Therefore, $\angle Q$ and $\angle Q'$ are supplementary.

Consequently, one conclusion that follows from the hypothesis is that

if $\overleftrightarrow{PQ}$ is shorter than $\overleftrightarrow{P'Q'}$ then $\angle Q$ and $\angle Q'$ are supplementary.

By a similar argument, another conclusion is that

if $\overleftrightarrow{P'Q'}$ is shorter than $\overleftrightarrow{PQ}$ then $\angle Q'$ and $\angle Q$ are supplementary.

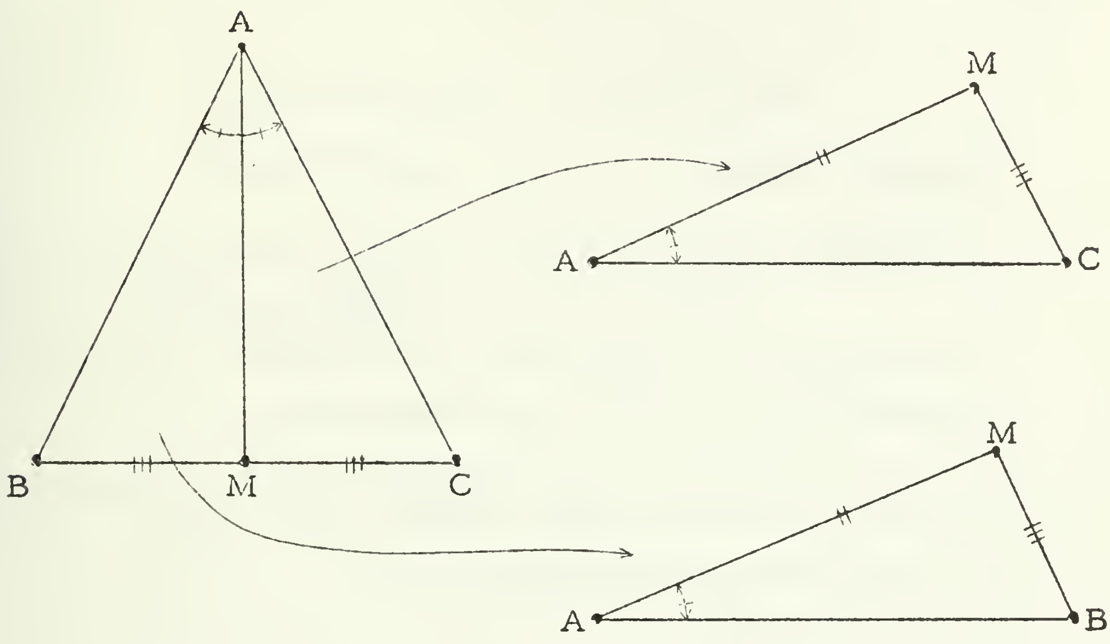
These conclusions can easily be combined into a third:

if $PQ \neq P'Q'$ then $\angle Q$ and $\angle Q'$ are supplementary,

which is equivalent to:

(*) if $\angle Q$ and $\angle Q'$ are not supplementary then $PQ = P'Q'$

Now, let's try to apply this result to the original problem of the angle bisector and the median. Here, the angles which we know to be



congruent are $\angle MAB$ and $\angle MAC$, and the pairs of congruent sides are $\overleftrightarrow{AM}$, $\overleftrightarrow{AM}$ and $\overleftrightarrow{MB}$, $\overleftrightarrow{MC}$. So, the conclusion is that if $\angle B$ and $\angle C$ are not supplementary then $AB = AC$. But, since $\angle B$ and $\angle C$ are angles of a triangle, it follows from Theorem 4-3 that $\angle B$ and $\angle C$ are not supplementary. Hence, $AB = AC$, and $\triangle ABC$ is isosceles.

Now, turn back to Exercise B1 on page 6-123 and Exercises B2 and B3 on page 6-123. In view of these exercises and the conclusion reached in the last paragraph, we have proved the following theorem:

<u>Theorem 4-12.</u>

(a) The median, altitude, and angle bisector from the vertex angle of an isosceles triangle are identical.

(b) If any two of the median, altitude, and angle bisector to a side of a triangle are identical then the triangle is isosceles with the given side as base.

EXERCISES

A. For all of the following exercises assume that, for ΔPQR and $\Delta P'Q'R'$, $\angle P \cong \angle P'$, $PR = P'R'$, and $RQ = R'Q'$. Then, for each exercise, add the assumption given in the exercise to these, and derive the following conclusion:

$$RPQ \longleftrightarrow R'P'Q' \text{ is a congruence}$$

Sample. $\angle Q$ and $\angle Q'$ are not supplementary

Solution. Since $\angle Q$ and $\angle Q'$ are not supplementary, it follows from (*) on page 6-126 that $PQ = P'Q'$. Since $RP = R'P'$ and $\angle P \cong \angle P'$, it follows from s.a.s. that $RPQ \longleftrightarrow R'P'Q'$ is a congruence.

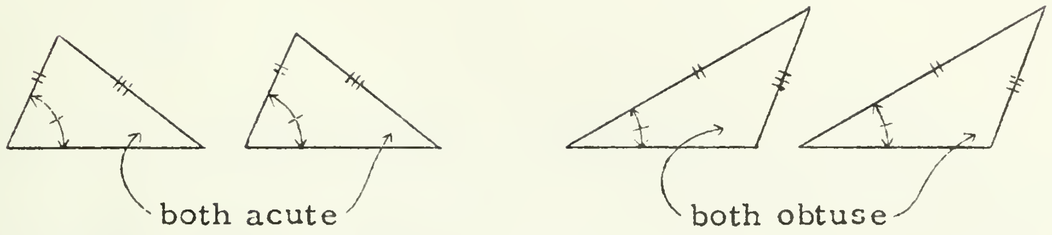
1. $\angle Q$ and $\angle Q'$ are acute angles [Hint. Build on the Sample.]
2. $\angle Q$ and $\angle Q'$ are obtuse angles
3. $\angle P$ is a right angle
4. $\angle P$ is not an acute angle
5. $\angle R$ and $\angle R'$ are obtuse angles

*

Your work in Part A establishes theorems that answer some questions which probably arose when you did the exercises in Parts C, D, and E starting on page 6-96. Go back and look at those exercises again.

In each of those exercises you were given a triangle and asked to find another which agreed with it in two sides and one of the nonincluded angles. You discovered that in some cases all such triangles seemed to be congruent to the given one. But, in other cases you could find triangles which satisfied the requirements but were not congruent to the given triangle. You may have noticed that this happened only when the given angle was an acute angle. In doing Exercise 4 of Part A above, you learned why.

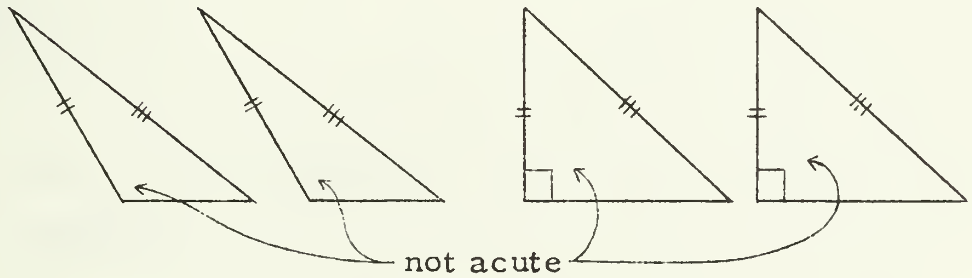
The result established in Exercise 4 is an easy consequence of Exercise 1 and Theorem 4-4. The results of Exercises 1 and 2 are readily combined into a single theorem:



Theorem 4-13.

If, for some matching of the vertices of one triangle with those of a second, two pairs of corresponding sides are congruent, the angles opposite the members of one of these pairs are congruent, and the angles opposite the members of the other pair are either both acute or both obtuse, then the matching is a congruence.

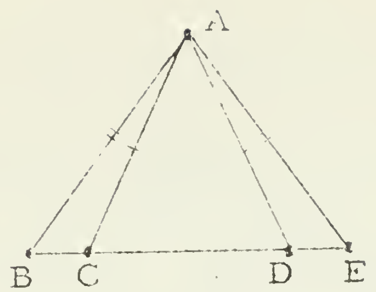
The result of Exercise 4 can also be stated as a theorem.



Theorem 4-14.

If, for some matching of the vertices of one triangle with those of a second, two pairs of corresponding sides are congruent, and the angles opposite the members of one of these pairs are congruent and are not acute, then the matching is a congruence.

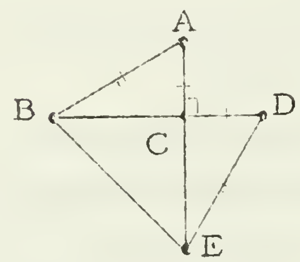
B. 1.



Hypothesis: $\triangle ABE$ and $\triangle ACD$ are isosceles triangles with vertex angles at A

Conclusion: $BC = DE$

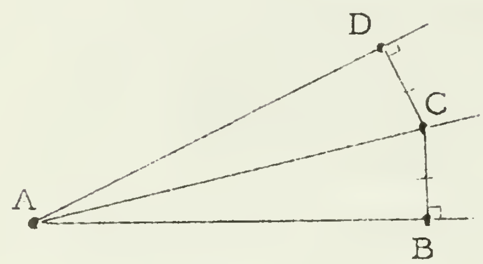
2.



Hypothesis: $\overleftrightarrow{AE} \perp \overleftrightarrow{BD}$ at C, $AC = CD$, $AB = DE$

Conclusion: $\triangle BCE$ is isosceles

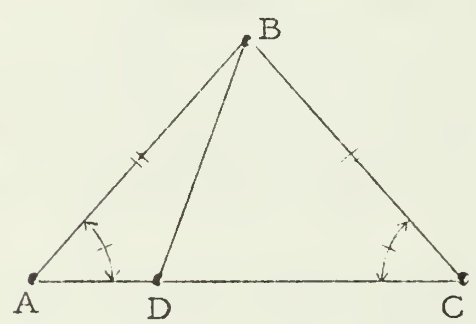
3.



Hypothesis: $CD = CB$, $\overleftrightarrow{CD} \perp \overleftrightarrow{AD}$ at D, $\overleftrightarrow{CB} \perp \overleftrightarrow{AB}$ at B

Conclusion: $\angle DAC \cong \angle BAC$

☆4.

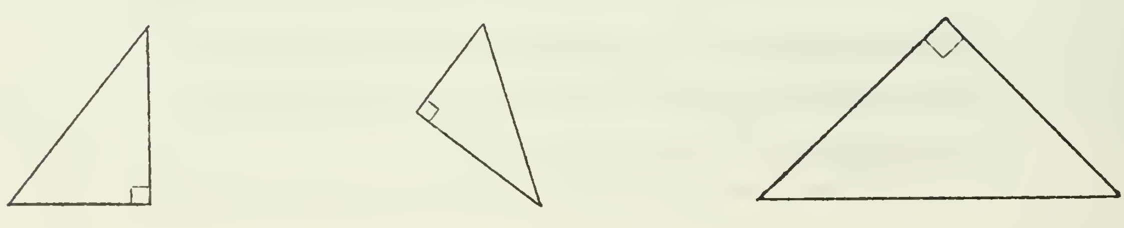


Hypothesis: $BA = BC$, $\angle A \cong \angle C$, $BAD \longleftrightarrow BCD$ is not a congruence

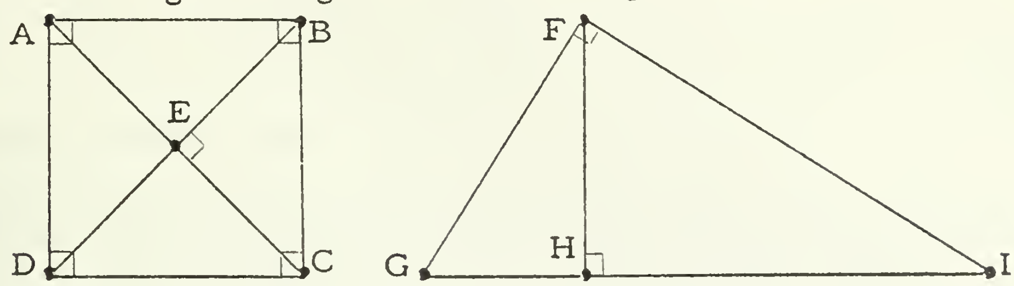
Conclusion: A, D, and C are collinear

* * *

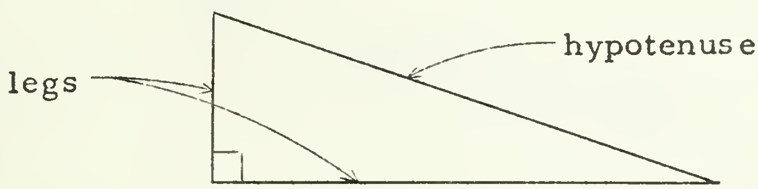
A triangle one of whose angles is a right angle is called <u>a right triangle</u>. [Can a right triangle be isosceles?] Here are pictures of several right triangles:



Name the right triangles in these two pictures:



The side opposite the right angle in a right triangle is called *the hypotenuse of the right triangle*. The other sides are called *legs*.



* * *

C. 1. Prove that the longest side of a right triangle is the hypotenuse.

2. Prove that the altitudes of a right triangle are *concurrent*. [That is, prove that the intersection of the three altitudes consists of just one point.]

3. Prove that no exterior angle of a right triangle is acute.

D. 1. Prove:

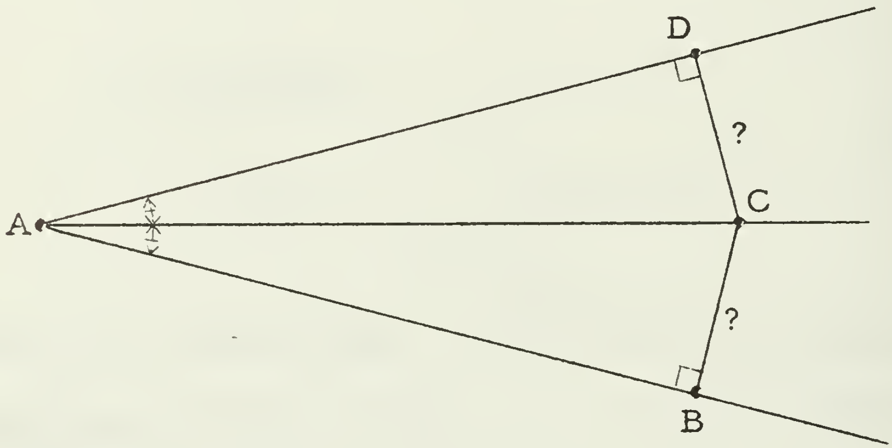
> *Theorem 4-15.* [The h.ℓ. theorem]
>
> If the hypotenuse and one leg of a right triangle are congruent to the hypotenuse and leg of a second right triangle then the matching of their vertices for which these are, respectively, corresponding sides is a congruence.

2. If the hypotenuse of one right triangle is congruent to a leg of another, can the two right triangles be congruent?

3. In the isosceles triangle, ΔABC, the vertex angle is $\angle A$. If the altitude $\overline{BD}$ from B is congruent to the altitude $\overline{CE}$ from C, show that $ACE \leftrightarrow ABD$ is a congruence.

* * *

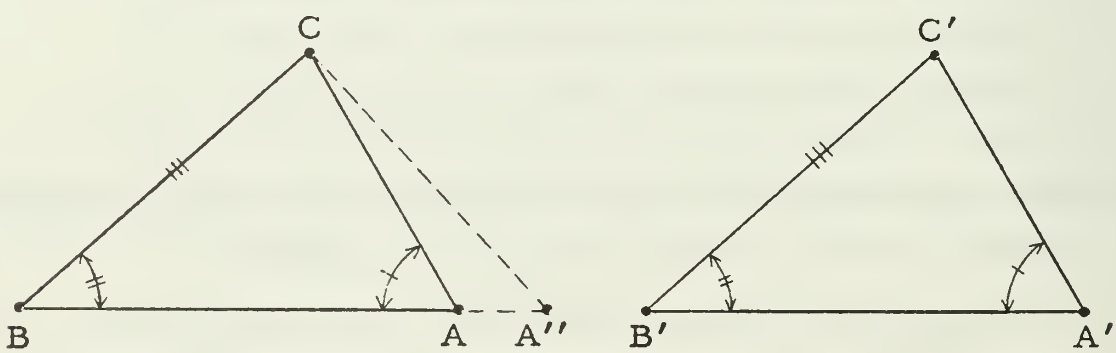
Turn to page 6-130 and look, again, at Exercise 3. What you proved there is that if a point is in the interior of the angle and equidistant from the sides [that is, from the lines containing the sides] then the point is on the bisector of the angle. It is natural to ask whether each point on the bisector of an angle is equidistant from the sides. [Clearly, we need consider only points of the bisector different from the vertex.] One way to show that this is the case is to show that



ACD $\longleftrightarrow$ ACB is a congruence. Let's see if we can do this.

We know that $\angle CDA \cong \angle CBA$ [Why?]. We also know that $\angle DAC \cong \angle BAC$ [Why?]. And, we know that AC = AC. So, the triangles agree in two angles and a side opposite one of the angles. These conditions do not meet the conditions of the a.s.a. theorem [or any of the other triangle-congruence theorems]. Perhaps there is an "a.a.s." theorem. Let's see.

Consider the triangles, $\triangle ABC$ and $\triangle A'B'C'$, for which $\angle A \cong \angle A'$,



$\angle B \cong \angle B'$, and BC = B'C'. Now, either BA = B'A' or $\overleftrightarrow{B'A'}$ is longer than $\overleftrightarrow{BA}$ or $\overleftrightarrow{BA}$ is longer than $\overleftrightarrow{B'A'}$.

By Axiom C, there is one and only one point A'' such that $A'' \in \overrightarrow{BA}$ and $BA'' = B'A'$. So, in view of the hypothesis, it follows from s.a.s. that $CBA'' \longleftrightarrow C'B'A'$ is a congruence. Hence $\angle A'' \cong \angle A'$, and, by hypothesis, $\angle CAB \cong \angle A''$.

Now, suppose $\overset{\longmapsto}{B'A'}$ is longer than $\overset{\longmapsto}{BA}$. Then, $\overset{\longmapsto}{BA''}$ is longer than $\overset{\longmapsto}{BA}$, and, by Theorem 2-6, $A \in \overline{BA''}$. In this case, $\angle CAB$ is an exterior angle of $\triangle CA''A$, and so, $\angle CAB \not\cong \angle A''$. Hence, $\overset{\longmapsto}{B'A'}$ is not longer than $\overset{\longmapsto}{BA}$.

Similarly, $\overset{\longmapsto}{BA}$ is not longer than $\overset{\longmapsto}{B'A'}$.

Hence, it follows from the hypothesis that $BA = B'A'$. And, furthermore, by either s.a.s. or a.s.a., $ABC \longleftrightarrow A'B'C'$ is a congruence.

Thus, we have proved another triangle-congruence theorem:

> <u>Theorem 4-16.</u> [a.a.s.]
>
> If, for some matching of the vertices of one triangle with those of a second, each of two angles and the side opposite the first of them is congruent to the corresponding part of the second triangle, then the matching is a congruence.

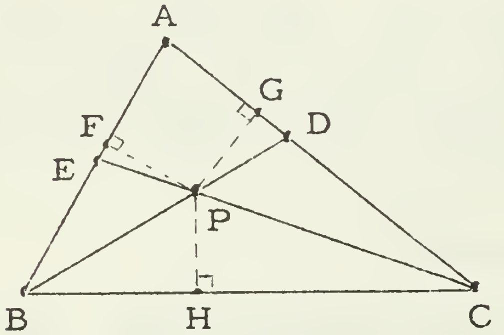
So, in view of Theorem 4-16, it follows that each point on the bisector of an angle is equidistant from the sides of the angle.

> <u>Theorem 4-17.</u>
>
> A point belongs to the bisector of an angle if and only if it is the vertex of the angle or is interior to the angle and equidistant from its sides.

EXERCISES

<u>A.</u> Show that two right triangles are congruent if they agree in the hypotenuse and an acute angle, or in a leg and the opposite angle.

B. By an Introduction Theorem, it can be shown that two angle bisectors of a triangle intersect at a point in the interior of the triangle.

1. 

Hypothesis: P is the point of intersection of the angle bisectors $\overrightarrow{BD}$ and $\overrightarrow{CE}$

Conclusion: P is equidistant from the sides of $\triangle ABC$

2. Prove that the point P of Exercise 1 belongs to the bisector of $\angle CAB$.

* * *

In the preceding exercises you proved the following theorem:

> **Theorem 4-18.**
> The angle bisectors of a triangle are concurrent in a point interior to the triangle.

C. It can be shown that the perpendicular bisectors of two sides of a triangle intersect in a point. Using this, prove:

> **Theorem 4-19.**
> The perpendicular bisectors of the sides of a triangle are concurrent.

D. Suppose lines ℓ and m intersect in a point P. Describe the set of points which are equidistant from ℓ and m.

☆ E. 1. Show that corresponding medians, angle bisectors, and altitudes of two congruent triangles are congruent.

2. Show that the altitudes to the legs of an isosceles triangle are congruent.

SUMMARY OF SECTION 6.04

Notation and Terminology

concurrent lines	[6-131]	triangle	[6-79]
distance between a		acute	[6-116]
point and a line	[6-119]	altitude of	[6-122]
exterior angle		angle bisector of	[6-122]
of a triangle	[6-115]	exterior angle of	[6-115]
angle opposite	[6-115]	median of	[6-122]
hypotenuse	[6-131]	obtuse	[6-116]
leg	[6-131]	right	[6-130]
perpendicular	[6-67]	right triangle	[6-130]
through a point	[6-120]	hypotenuse of	[6-131]
foot of	[6-119]	leg of	[6-131]

$\overline{AB}$ is longer than $\overline{CD}$	$\angle ABC$ is larger than $\angle DEF$
$AB > CD$	$m(\angle ABC) > m(\angle DEF)$
$\overline{AB}$ is shorter than $\overline{CD}$	$\angle ABC$ is smaller than $\angle DEF$
$AB < CD$	$m(\angle ABC) < m(\angle DEF)$

Theorems

4-1. The sum of two sides of a triangle is greater than the third.

4-2. The sum of the measures of two angles of a triangle is less than 180.

4-3. No two angles of a triangle are supplementary.

4-4. If one angle of a triangle is a right angle or an obtuse angle then the others are acute angles.

4-5. An exterior angle of a triangle is larger than each of the angles opposite it.

4-6. If one side of a triangle is longer than another then the angle opposite the first is larger than the angle opposite the second.

4-7. If one angle of a triangle is larger than another then the side opposite the first is longer than the side opposite the second.

4-8. [Definition of perpendicular through a point]

$\forall_m \forall_\ell \forall_P$

m is the perpendicular to ℓ through P.

if and only if

$P \in m$ and $m \cup \ell$ contains a right angle

4-9. The shortest segment from a point to a line is the one which is perpendicular to the line.

4-10. For each ΔXYZ and each $W \in \overline{YZ}$ if $\overleftrightarrow{XY}$ is not longer than $\overleftrightarrow{XZ}$ then $\overleftrightarrow{XZ}$ is longer than $\overleftrightarrow{XW}$.

4-11. If two triangles agree in two pairs of sides but not in the included angles then the triangle with the larger included angle has the longer third side.

4-12. (a) The median, altitude, and angle bisector from the vertex angle of an isosceles triangle are identical.

(b) If any two of the median, altitude, and angle bisector to a side of a triangle are identical then the triangle is isosceles with the given side as base.

4-13. If, for some matching of the vertices of one triangle with those of a second, two pairs of corresponding sides are congruent, the angles opposite the members of one of these pairs are congruent, and the angles opposite the members of the other pair are either both acute or both obtuse, then the matching is a congruence.

4-14. If, for some matching of the vertices of one triangle with those of a second, two pairs of corresponding sides are congruent, and the angles opposite the members of one of these pairs are congruent and are not acute, then the matching is a congruence.

4-15. If the hypotenuse and one leg of a right triangle are congruent to the hypotenuse and leg of a second right triangle then the matching of their vertices for which these are, respectively, corresponding sides is a congruence. [The h.ℓ.theorem]

4-16. If, for some matching of the vertices of one triangle with those of a second, each of two angles and the side opposite the first of them is congruent to the corresponding part of the second triangle, then the matching is a congruence. [a.a.s.]

4-17. A point belongs to the bisector of an angle if and only if it is the vertex of the angle or is interior to the angle and equidistant from its sides.

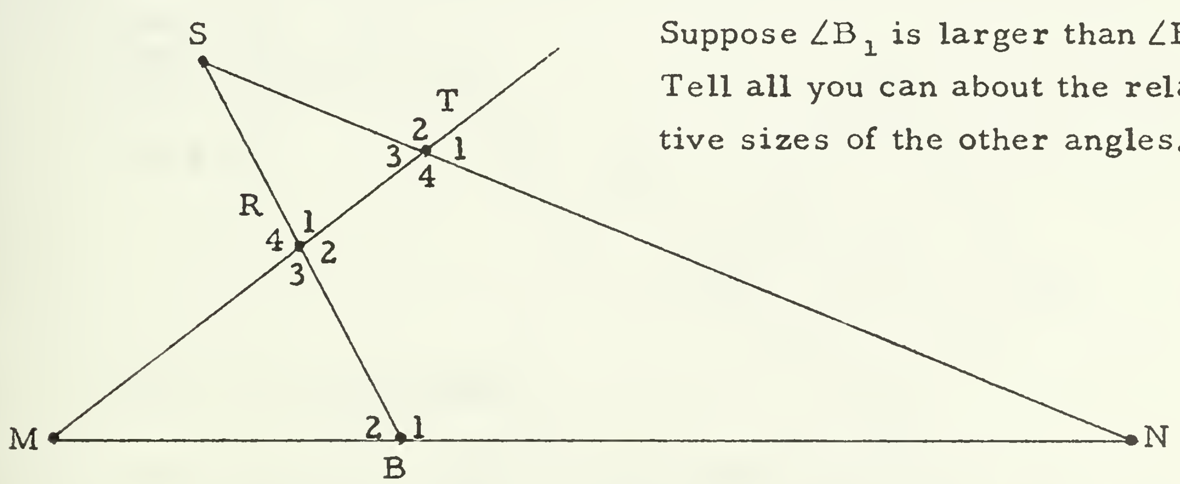
4-18. The angle bisectors of a triangle are concurrent in a point interior to the triangle.

4-19. The perpendicular bisectors of the sides of a triangle are concurrent.

MISCELLANEOUS EXERCISES

1. If, in ΔMPQ, $m(\angle P) = 60$ and $MQ > MP$, then $m(\angle Q)$ ____________.

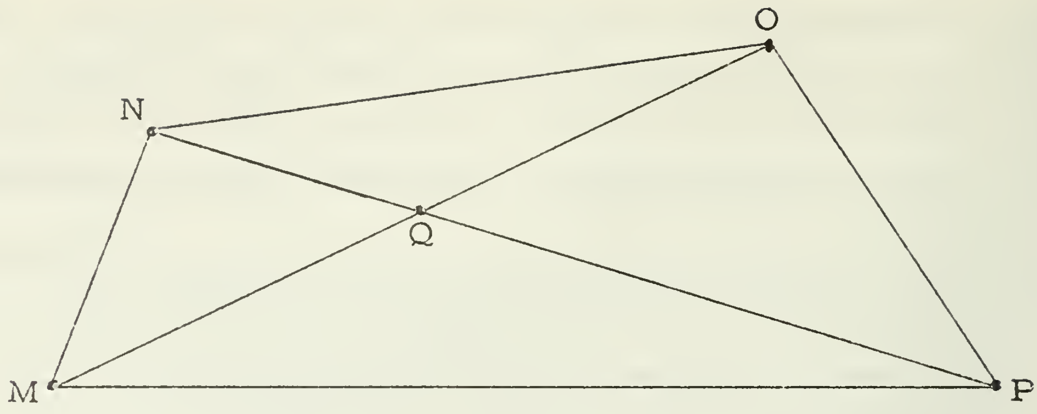
2. If, in ΔXYZ, $XY = 4$ and $YZ = 9$, then XZ ____________________.

3. 

Suppose $\angle B_1$ is larger than $\angle B_2$. Tell all you can about the relative sizes of the other angles.

4. Prove that if $\overleftrightarrow{AD}$ is the angle bisector from A of ΔABC then $AC > CD$ and $AB > BD$.

5.



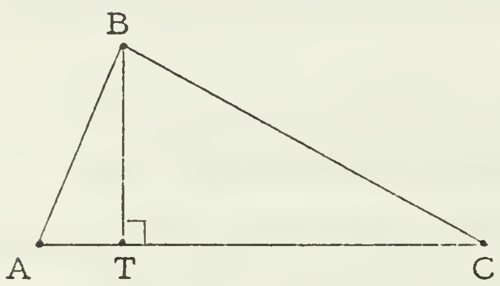
(a) What angles are smaller than $\angle NQM$?

(b) What angles are smaller than $\angle MQP$?

(c) Suppose $MQ = NO$. Tell all you can about the relative sizes of the other segments.

6. Prove that the base angles of an isosceles triangle are acute.

7.

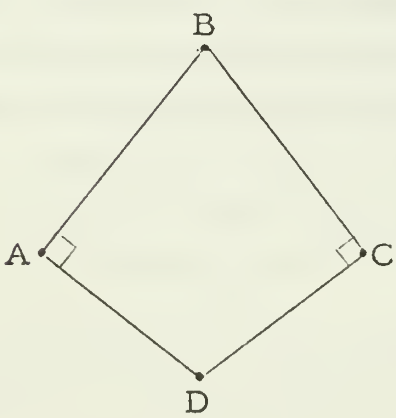


<u>Hypothesis:</u> $\overleftrightarrow{BT} \perp \overleftrightarrow{AC}$,
$AT < TC$

<u>Conclusion:</u> $AB < BC$

[<u>Hint.</u> Locate $P \in \overline{TC}$ such that PT T

8.



<u>Hypothesis:</u> $\overleftrightarrow{BA} \perp \overleftrightarrow{AD}$,
$\overleftrightarrow{BC} \perp \overleftrightarrow{CD}$,
$AB = BC$

<u>Conclusion:</u> $AD = DC$

9. The measures of the sides of a right triangle are 30, 40, and 50. What is the measure of the hypotenuse?

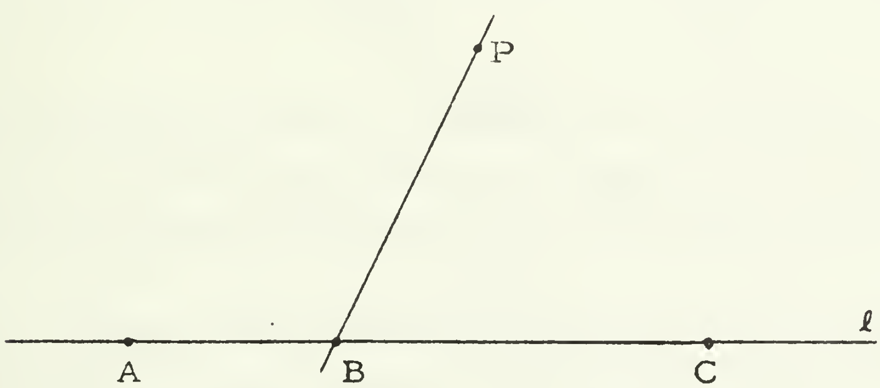
[Supplementary exercises are on page 6-423.]

6.05 Parallel lines.--In the Introduction you learned that ℓ is parallel to m [for short: $\ell \,||\, m$] if and only if $\ell \cap m = \emptyset$. One of the Introduction Axioms, Axiom 4, says that if each of two lines is parallel to a third then the two lines are parallel to each other. From this it was easy to prove an Introduction theorem:

> Theorem 3. For each line ℓ, and each point $P \notin \ell$, there is at most one line which contains P and is parallel to ℓ.

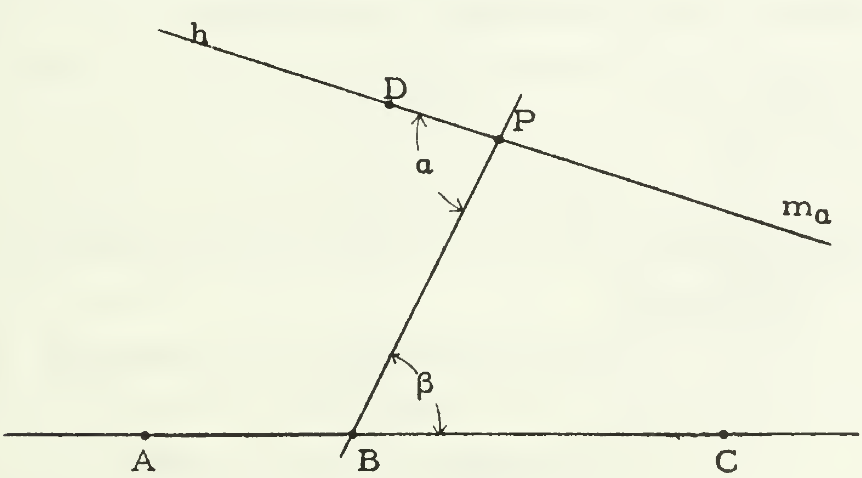
So, if ℓ is a line and $P \notin \ell$, we know that there are not two lines through P and parallel to ℓ. But, is there even one such line? To answer this question let's consider all the lines through P and try to prove that one of them is parallel to ℓ.

Let A, B, and C be points on ℓ such that $B \in \overline{AC}$. $\overleftrightarrow{PB}$ separates



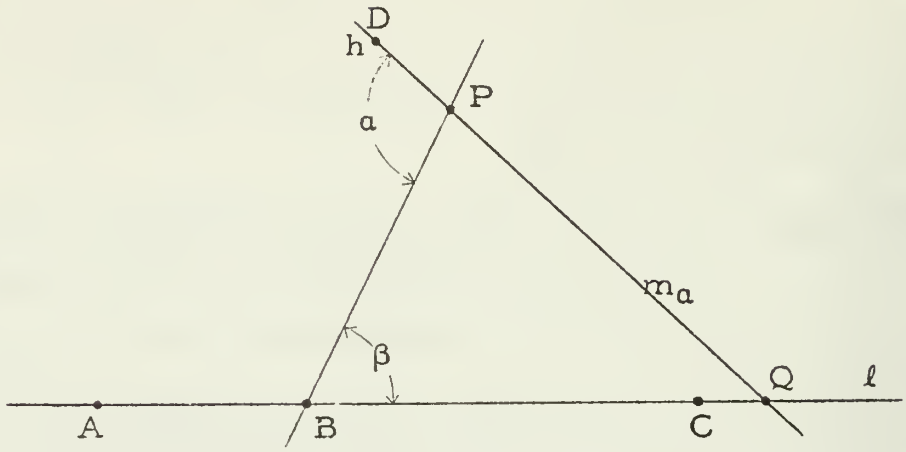
the plane into two half-planes, the A-side of $\overleftrightarrow{PB}$ and the C-side of $\overleftrightarrow{PB}$.

Axiom E tells us that for each α between 0 and 180, there is one [and only one] half-line h with vertex P, and contained in the A-side of $\overleftrightarrow{PB}$, such that $m(h \cup \overrightarrow{PB}) = \alpha$.

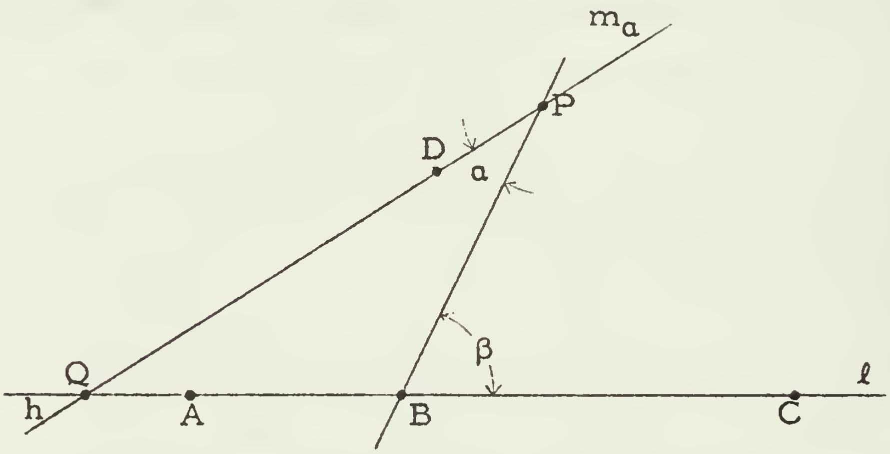


Let $D \in h$. So, $m(\angle DPB) = \alpha$. The half-line h is a subset of one and only one line. Let that line be m_α, and let $m(\angle PBC) = \beta$. If m_α intersects ℓ in a point Q then $Q \neq B$ [Why?]. So, either $Q \in \overrightarrow{BC}$ or $Q \in \overrightarrow{BA}$.

If $Q \in \overrightarrow{BC}$ then $\angle DPB$ is an exterior angle of $\triangle PBQ$; so, $\alpha > \beta$.



If $Q \in \overrightarrow{BA}$ then $\angle PBC$ is an exterior angle of $\triangle PBQ$; so, $\alpha < \beta$.



Thus, if m_α intersects ℓ then $\alpha \neq \beta$. So [contrapositively], if $\alpha = \beta$ then m_α does not intersect ℓ. Hence, there is at least one line through P and parallel to ℓ. This is the line for which $\alpha = \beta$. We shall call it 'm_β'.

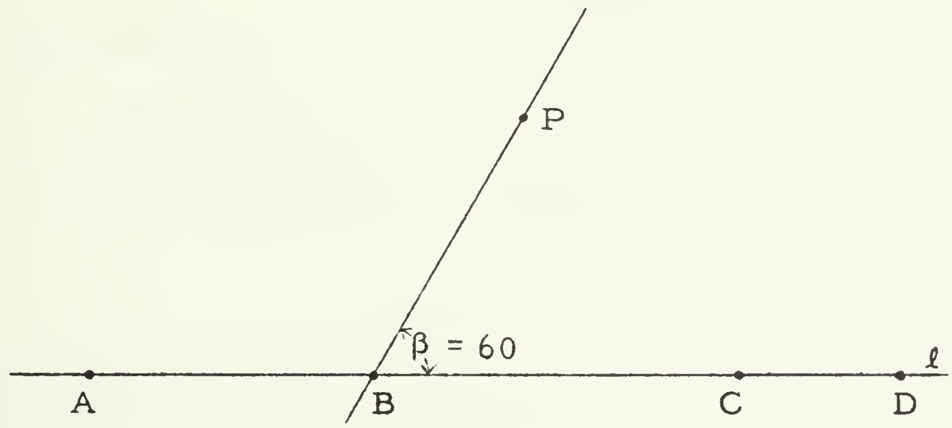
Combining this result with Theorem 3 quoted above, we see that there is one and only one line through P and parallel to ℓ. So, when $P \notin \ell$, we can speak of <u>the</u> line through P parallel to ℓ. Thus, we have the following theorem:

> <u>Theorem 5-1.</u> [Def. of parallel through a point]
>
> $\forall_m \forall_\ell \forall_{P \notin \ell}$
>
> m is the parallel to ℓ through P
> if and only if
> $P \in m$ and $m \cap \ell = \emptyset$

Notice that the proof given above shows that $P \in m_\beta$ and $m_\beta \cap \ell = \emptyset$. So, the line through P parallel to ℓ is m_β.

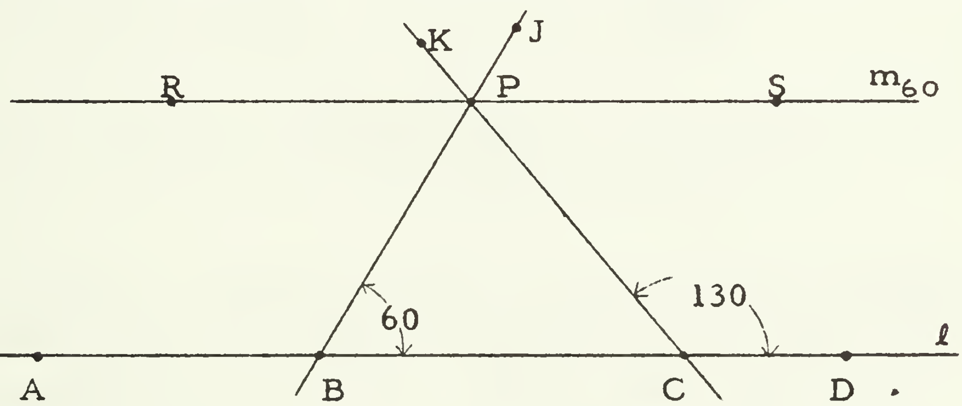
EXERCISES

A. Here is a picture of a line ℓ and a point $P \notin \ell$. Use your straightedge and either a compass or a protractor to draw m_{60}. Is $m_{60} \parallel \ell$? Does $m_{59.9}$ intersect ℓ?



B. In the picture for Part A, $m(\angle PCD) = 130$. Find a point R on the A-side of $\overleftrightarrow{PC}$ such that $\angle RPC$ is an angle of $130°$. [Did you need to use any drawing instruments to do this? Why not?]

C. Here is another picture of ℓ and P, and the line m_{60} which you drew in Part A and a point R such as you found in Part B.



(a) $m(\angle RPB) =$ _______ (b) $m(\angle RPC) =$ _______

(c) $m(\angle SPC) =$ _______ (d) $m(\angle KPR) =$ _______

(e) $m(\angle JPS) =$ _______ (f) $m(\angle KPS) =$ _______

(g) $m(\angle SPB) =$ _______ (h) $m(\angle KPJ) =$ _______

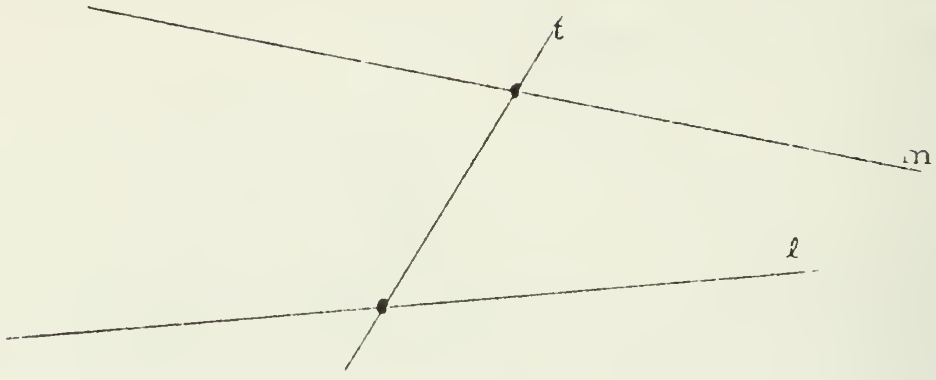
(i) $m(\angle CBP) + m(\angle BPC) + m(\angle PCB) =$ _______

(j) $m(\angle CBP) + m(\angle PCB) =$ _______

(k) $m(\angle JBC) =$ _______ (ℓ) $m(\angle KPB) =$ _______

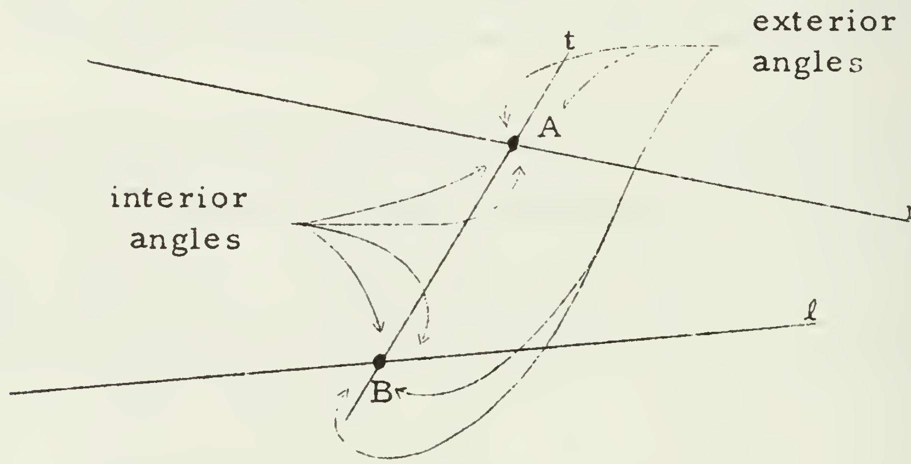
TRANSVERSALS

Suppose ℓ and m are two lines. A line t which intersects $\ell \cup m$



in exactly two points is said to be a <u>transversal</u> of ℓ and m. [Draw a picture of a line which crosses each of two other lines but is not a transversal of these lines.]

Suppose $t \cap (\ell \cup m) = \{A, B\}$, where $A \neq B$. There are precisely



eight angles each of which is a subset of either $t \cup m$ or $t \cup \ell$. Look at the four angles each of which is a subset of $t \cup m$. Do you see that two of them contain both A and B? That two of them contain A but do not contain B? Do the same for the angles which are subsets of $t \cup \ell$. The four angles which contain both A and B are called <u>interior angles</u>. The other four are called <u>exterior angles</u>.

Each pair of interior angles which have different vertices and contain points on opposite sides of t is called <u>a pair of alternate interior angles</u>. [How many such pairs are there?] Do you see that for each pair of alternate interior angles, one of them lies in one of the closed half-planes determined by the transversal while the other one lies in

the opposite closed half-plane. Give a description of a pair of alternate exterior angles.

Pairs of interior angles [or exterior angles] which have different vertices but are not alternate angles are called consecutive interior angles [or consecutive exterior angles]. Do you see that both of a pair of consecutive interior angles lie in the same closed half-plane determined by the transversal?

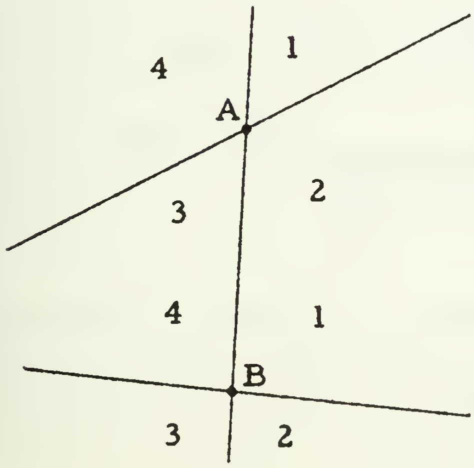
A pair consisting of an interior angle and an exterior angle which have different vertices and are contained in the same closed half-plane determined by the transversal is called a pair of corresponding angles.

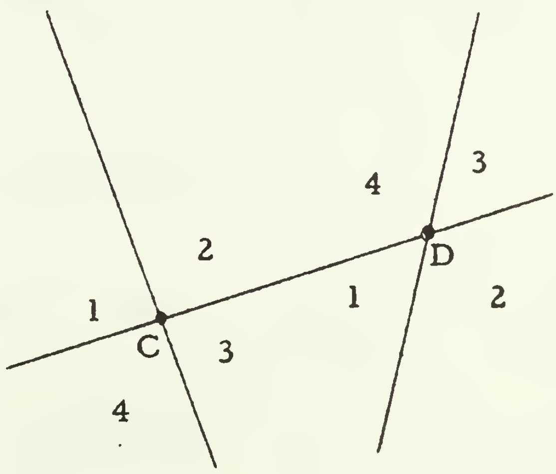
EXERCISES

A. For each of the following figures, list pairs of angles of each of the following types.

(1) alternate interior angles
(2) corresponding angles
(3) alternate exterior angles
(4) consecutive interior angles
(5) consecutive exterior angles
(6) adjacent angles
(7) vertical angles

[The angles are numbered for easy reference. For example, you can say that $\angle A_1$ and $\angle B_1$ are corresponding angles.]





* * *

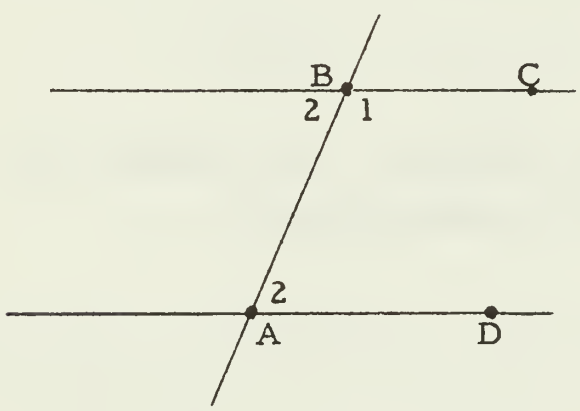
Turn back to page 6-140. In the discussion leading to Theorem 5-1, the line $\overleftrightarrow{PB}$ is a transversal of m_α and ℓ. The angles $\angle DPB$ and $\angle PBC$ are alternate interior angles. So, in proving that m_β is a line through P parallel to ℓ, we proved:

> Theorem 5-2.
>
> If two lines are cut by a transversal in such a way that some two alternate interior angles are congruent then the two lines are parallel.

Since [by Theorem 3] it follows that m_β is the only line through P parallel to ℓ, we also proved:

> Theorem 5-3.
>
> If two parallel lines are cut by a transversal then each two alternate interior angles are congruent.

Example.



Hypothesis: $\angle B_1$ and $\angle A_2$ are supplementary consecutive interior angles .

Conclusion: $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$

Plan. We can show that $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$ if we can show that the alternate interior angles, $\angle A_2$ and $\angle B_2$, are congruent. Now, the hypothesis tells us that $\angle A_2$ is the supplement of $\angle B_1$. So, to show that $\angle A_2 \cong \angle B_2$ just show that $\angle B_2$ is also a supplement of $\angle B_1$.

Solution.

(1) $\angle B_1$ and $\angle A_2$ are supplementary [Hypothesis]

(2) $\angle B_1$ and $\angle B_2$ are adjacent angles whose noncommon sides are collinear [figure; def. of adjacent angles]

(3) Adjacent angles are supplementary if and only if their noncommon sides are collinear. [theorem]

(4) $\angle B_1$ and $\angle B_2$ are supplementary [(2) and the if-part of (3)]

(5) Supplements of the same angle or of congruent angles are congruent. [theorem]

(6) $\angle B_2 \cong \angle A_2$ [(1), (4), and (5)]

(7) $\angle B_2$ and $\angle A_2$ are alternate interior angles [figure; def. of alt. int. $\angle$s]

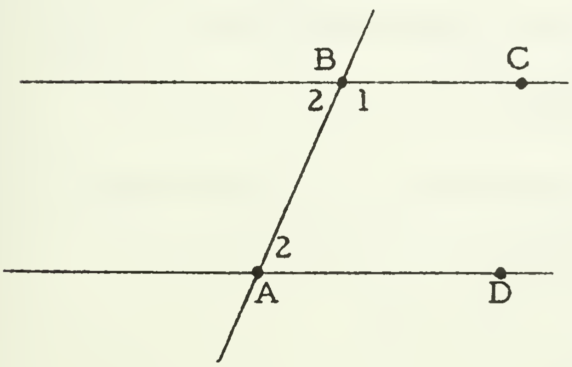
(8) If two lines are cut by a transversal in such a way that some two alternate interior angles are congruent then the two lines are parallel. [theorem]

(9) $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$ [(7) and (8)]

This example gives us the following theorem:

> Theorem 5-4.
>
> If two lines are cut by a transversal in such a way that some two consecutive interior angles are supplementary then the two lines are parallel.

B.



Hypothesis: $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$, $\angle B_1$ and $\angle A_2$ are consecutive interior angles

Conclusion: $\angle B_1$ and $\angle A_2$ are supplementary

C. 1. In Part B you proved:

> Theorem 5-5.
>
> If two parallel lines are cut by a transversal then each two consecutive interior angles are supplementary.

This theorem on consecutive interior angles is analogous to Theorem 5-3 [Why?]. In the box below write a theorem about corresponding angles which is analogous to Theorem 5-2 and 5-4. Then, carry out an Hypothesis-Conclusion argument to justify the theorem.

> Theorem 5-6.

2. In the box below write a theorem about corresponding angles which is analogous to Theorems 5-3 and 5-5. Then carry out an Hypothesis-Conclusion argument to justify the theorem.

> Theorem 5-7.

D. Carry out Hypothesis-Conclusion arguments to justify the following theorems.

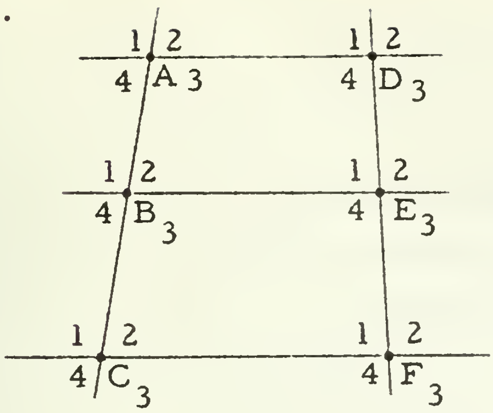
1.

> Theorem 5-8.
>
> Two lines which are perpendicular to a third line are parallel.

2.

> Theorem 5-9.
>
> If one of two parallel lines is perpendicular to a third line then so is the other.

<u>E.</u> 1.

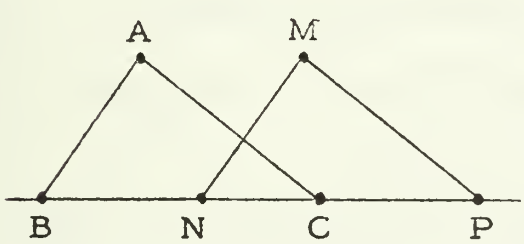


<u>Hypothesis:</u> $\angle D_3 \cong \angle E_1$, $\angle E_1 \cong \angle F_1$

<u>Conclusion:</u> $\angle A_2$ and $\angle C_3$ are supplementary

2. Given that $\triangle ABC$ is isosceles with $\angle A$ as vertex angle. Line ℓ is parallel to $\overleftrightarrow{AB}$ and intersects $\overline{AC}$ in P and $\overline{BC}$ in Q. Prove that $\triangle PCQ$ is isosceles. [Does the conclusion still follow if $\angle C$, rather than $\angle A$, is the vertex angle of $\triangle ABC$?]

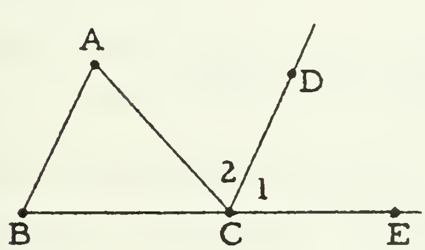
3.



<u>Hypothesis:</u> $\overleftrightarrow{AB} \parallel \overleftrightarrow{MN}$, $\overline{AB} \cong \overline{MN}$, $\overline{BN} \cong \overline{CP}$

<u>Conclusion:</u> $\overleftrightarrow{AC} \parallel \overleftrightarrow{MP}$

<u>F.</u> 1.



<u>Hypothesis:</u> $C \in \overline{BE}$, $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$

<u>Conclusion:</u> $m(\angle A) + m(\angle B) = m(\angle ACE)$

2. Same Hypothesis as in Exercise 1. Prove that

$$m(\angle A) + m(\angle B) + m(\angle ACB) = 180.$$

* * *

Part F gives us two interesting theorems:

> <u>Theorem 5-10.</u>
> The measure of an exterior angle of a triangle is the sum of the measures of the two opposite angles of the triangle.

> <u>Theorem 5-11.</u>
> The sum of the measures of the angles of a triangle is 180.

* * *

3. Theorem 5-11 has many interesting consequences. For example, given the measures of two angles of a triangle, we can deduce the measure of the third.

   (a) If $m(\angle A) = 40$ and $m(\angle B) = 70$ then $m(\angle C)$ of $\triangle ABC$ is ______ .

   (b) If $m(\angle A) = 2 \cdot m(\angle B)$, and $m(\angle C)$ of $\triangle ABC$ is 120, then $m(\angle A) =$ ______ and $m(\angle B) =$ ______ .

   (c) The sum of the measures of the acute angles of a right triangle is ______ ; so, the acute angles are ______ .

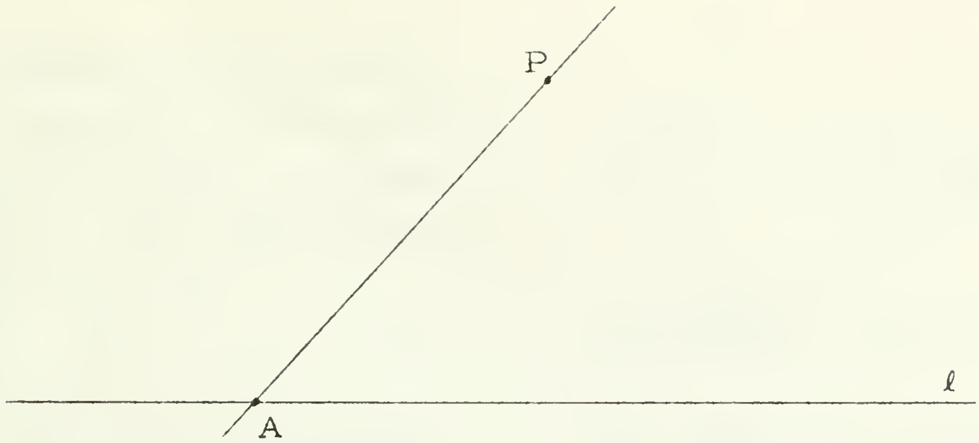
   (d) Each angle of an equilateral triangle is an angle of ______ degrees.

   (e) If one angle of an isosceles triangle is an angle of 45°, what are the measures of the others?

   (f) Suppose G is the point of concurrence of the bisectors of the angles of equilateral triangle, $\triangle ABC$. What is the measure of $\angle AGB$?

   (g) Suppose $m(\angle A) = a$. If K is the point of concurrence of the bisectors of the angles of $\triangle ABC$ then $m(\angle BKC) =$ ______ .

<u>G.</u> One of the ways of drawing a line parallel to a given line ℓ through a point $P \notin \ell$ is to use a protractor or a compass and apply Theorem 5-2. Here is another method.



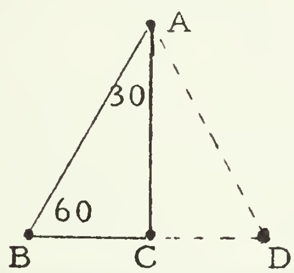
(1) Locate the midpoint of $\overline{PA}$. Call it 'M'.

(2) Draw a line through M other than $\overleftrightarrow{PA}$ which intersects ℓ in a point N.

(3) Locate the point Q on $\overrightarrow{NM}$ such that NM = MQ.

(4) The line $\overleftrightarrow{PQ}$ is parallel to ℓ. <u>Prove it</u>.

<u>H.</u> 1.



<u>Hypothesis</u>: $\angle A$ is an angle of 30°, $\angle B$ is an angle of 60°

<u>Conclusion</u>: $BC = \frac{1}{2} \cdot AB$

[<u>Hint.</u> There is a point $D \in \overrightarrow{BC}$ such that BC = CD.]

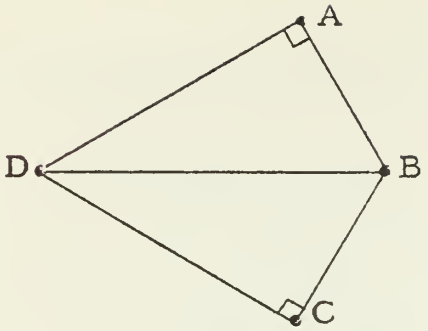
*

Exercise 1 gives us the following theorem:

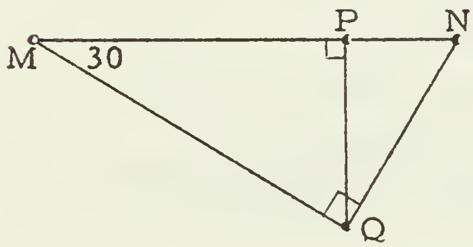
> <u>Theorem 5-12.</u>
> The measure of the shorter leg of a 30-60-90 triangle is half the measure of the hypotenuse.

*

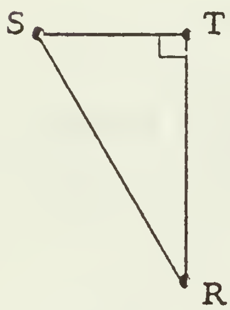
2. In $\triangle ABC$ and $\triangle A'B'C'$, $\angle C$ and $\angle C'$ are right angles and $AC = A'C'$, $m(\angle B) = 60$, and $m(\angle B') = 55$. Arrange the five segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$, $\overline{A'B'}$, and $\overline{B'C'}$, in order of size starting with the largest.

3. 

Suppose the right triangles $\triangle DAB$ and $\triangle DCB$ are congruent, and $\angle ADC$ is an angle of $60°$. Show that $AB + BC \doteq DB$.

4. 

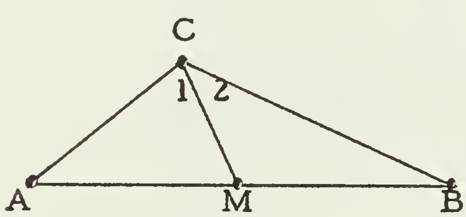
Suppose $\overleftrightarrow{NQ} \perp \overleftrightarrow{MQ}$ and $\overleftrightarrow{QP} \perp \overleftrightarrow{MN}$ and $\angle M$ is an angle of $30°$. What fraction of MN is PN?

5. 

Suppose $ST = \frac{1}{2} \cdot SR$ and $\overleftrightarrow{RT} \perp \overleftrightarrow{ST}$. What is the measure of $\angle RST$? Prove it.

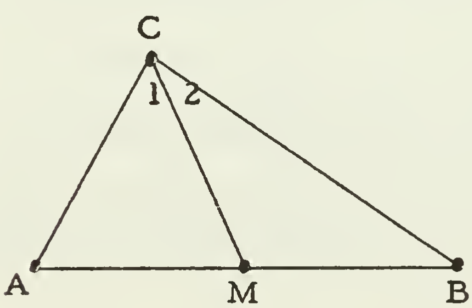
6. (a) Show that if the measure of the median to a side of a triangle is half the measure of that side, then the triangle is a right triangle with the given side as hypotenuse.

(b) Can a triangle have two medians such that the measure of each is half the measure of the side to which it is drawn?

7. 

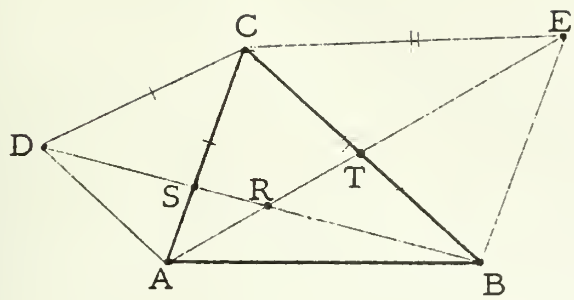
Hypothesis: M is the midpoint of $\overleftrightarrow{AB}$, $CM < \frac{1}{2} \cdot AB$

Conclusion: $\angle ACB$ is obtuse

8. 

Hypothesis: M is the midpoint of $\overleftrightarrow{AB}$, $CM > \frac{1}{2} \cdot AB$

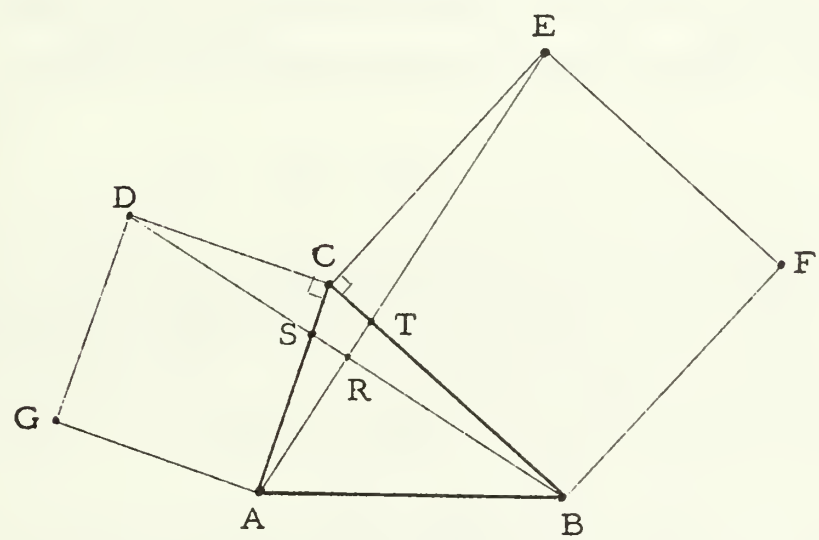
Conclusion: $\angle ACB$ is acute

9. 

Hypothesis: $\angle BCE \cong \angle ACD$,
$CB = CE$,
$AC = CD$

Conclusion: $\triangle DBC \cong \triangle AEC$

If $m(\angle BCE) = \alpha$ and $m(\angle CEA) = \beta$,
then $m(\angle ETB) =$ __________. [Why?]
But, $m(\angle CBD) =$ __________. [Why?]
So, $m(\angle TRB) =$ __________.

10. 

Suppose $m(\angle ECB) = 90$,
$m(\angle ACD) = 90$,
$AC = DC$,
$CE = CB$.

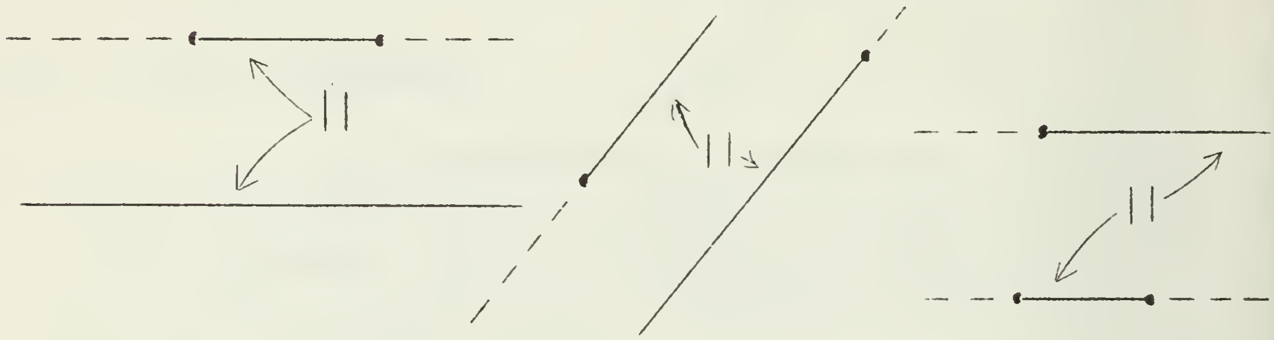
Find $m(\angle TRS)$.

☆11. Exercises 9 and 10 have certain features in common. Try to write an exercise analogous to these so that the figures which correspond to $\triangle BCE$ and $\triangle ACD$ in Exercise 9 and to the "four-sided" figures BCEF and ACDG in Exercise 10 have five sides.

[Supplementary exercises are on page 6-424.]

* * *

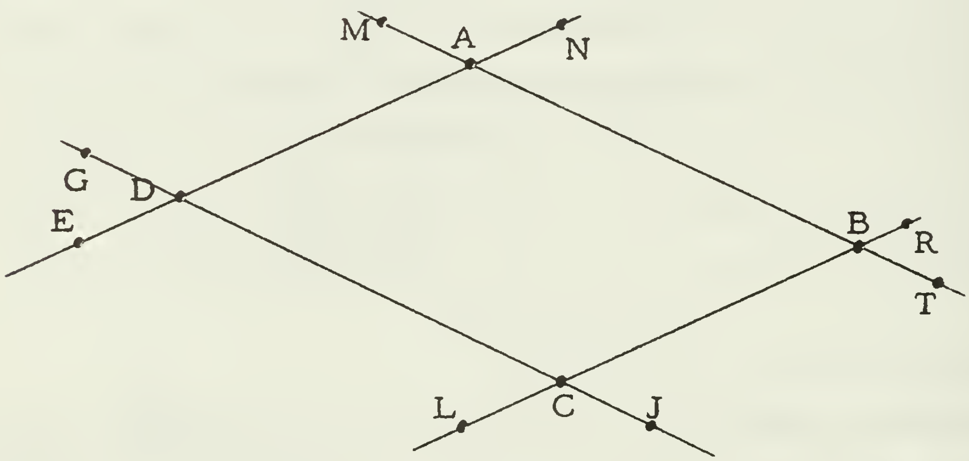
Just as in the case of perpendicularity where we spoke of segments and other subsets of lines being perpendicular when the lines determined by them are perpendicular, so we will speak of segments, half-lines, and rays being parallel if they are subsets of parallel lines.



[If the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel then $\overline{AB} \parallel \overline{CD}$, $\overrightarrow{AB} \parallel \overrightarrow{DC}$, $\overrightarrow{AB} \parallel \overrightarrow{CD}$, etc. If $\overline{MN} \cap \overline{PQ} = \emptyset$, does it follow that $\overline{MN} \parallel \overline{PQ}$?]

* * *

I. In the figure below, lines $\overleftrightarrow{EN}$ and $\overleftrightarrow{LR}$ are parallel, and lines $\overleftrightarrow{GJ}$ and $\overleftrightarrow{MT}$ are parallel. Notice that the rays $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are parallel [Why?] So are the rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$, but these rays "point" in opposite directions whereas the rays $\overrightarrow{AB}$ and $\overrightarrow{DC}$ point in the same direction.



1. Which of the rays listed below are parallel to $\overrightarrow{AB}$?
$\overrightarrow{GD}$, $\overrightarrow{DE}$, $\overrightarrow{DC}$, $\overrightarrow{JC}$, $\overrightarrow{CJ}$, $\overrightarrow{CB}$, $\overrightarrow{TM}$, $\overrightarrow{BT}$

2. Which of these rays point in the same direction as $\overrightarrow{AB}$?
$\overrightarrow{GD}$, $\overrightarrow{DG}$, $\overrightarrow{DJ}$, $\overrightarrow{AB}$, $\overrightarrow{AT}$, $\overrightarrow{TM}$, $\overrightarrow{BT}$

3. Which of these rays point in the opposite direction from $\overrightarrow{AB}$?
$\overrightarrow{GD}$, $\overrightarrow{DG}$, $\overrightarrow{AE}$, $\overrightarrow{EA}$, $\overrightarrow{AM}$, $\overrightarrow{LC}$, $\overrightarrow{BA}$

4. (a) Do rays $\overrightarrow{AN}$ and $\overrightarrow{CR}$ point in the same direction?
   (b) Do rays $\overrightarrow{AT}$ and $\overrightarrow{CJ}$ point in the same direction?
   (c) What can you say about the angles, $\overrightarrow{AN} \cup \overrightarrow{AT}$ and $\overrightarrow{CR} \cup \overrightarrow{CJ}$?

5. (a) Do rays $\overrightarrow{AD}$ and $\overrightarrow{CB}$ point in opposite directions?
   (b) Do rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ point in opposite directions?
   (c) What can you say about $\overrightarrow{AD} \cup \overrightarrow{AB}$ and $\overrightarrow{CB} \cup \overrightarrow{CD}$?

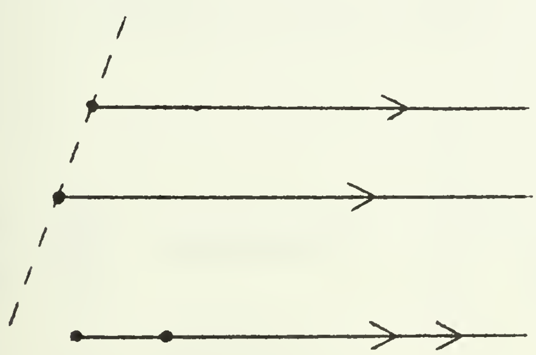
6. (a) Name a pair of angles in the figure -- one with vertex D and the other with vertex C -- such that if you match their sides, the corresponding sides are rays which point in the same direction. What can you say about these angles?
   (b) Repeat (a) for corresponding sides pointing in opposite directions.

7. Consider the angle $\overrightarrow{AM} \cup \overrightarrow{AD}$. Name another angle in the figure such that one of its sides points in the same direction as $\overrightarrow{AM}$ and the other points in the opposite direction from $\overrightarrow{AD}$. What can you say about these angles?

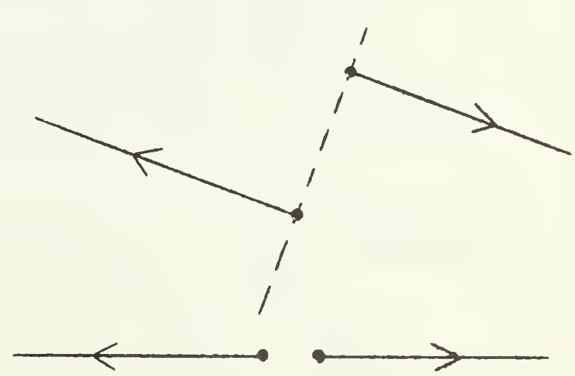
*

Two rays are <u>similarly directed</u> if and only if <u>either</u> they are parallel and are contained in the same closed half-plane determined by the line through their vertices <u>or</u> one is a subset of the other.

Also, two rays are <u>oppositely directed</u> if and only if either they are parallel and are contained in opposite closed half-planes determined by the line through their vertices or are collinear and neither is a subset of the other.



<u>similarly directed</u>



<u>oppositely directed</u>

The preceding definitions and exercises suggest the following theorem:

> <u>Theorem 5-13.</u>
> If the sides of two angles can be matched in such a way that corresponding sides are similarly directed then the angles are congruent.

What are the cases you need to treat in order to justify this theorem?

*

8. In the box below, write another theorem on congruent angles analogous to Theorem 5-13, which is also justified by your work in the preceding exercises.

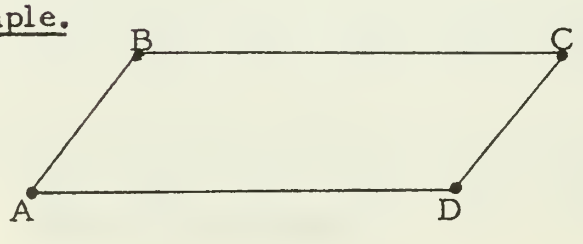
> <u>Theorem 5-14.</u>

9. Draw a picture to illustrate the situation described in the following theorem, and complete the theorem.

> <u>Theorem 5-15.</u>
> If the sides of two angles can be matched in such a way that two corresponding sides are similarly directed and the other two corresponding sides are oppositely directed then the angles are ____________________ .

* * *

<u>Example.</u>



<u>Hypothesis:</u> $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$

<u>Conclusion:</u> $\angle B \cong \angle D$, $\angle A \cong \angle C$

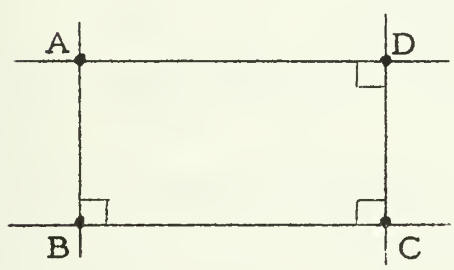
Plan. We can show that $\angle B \cong \angle D$ if we can find a matching such that the corresponding sides are similarly or oppositely directed. From the hypothesis we can conclude that $\overrightarrow{BC} \parallel \overrightarrow{DA}$ and $\overrightarrow{BA} \parallel \overrightarrow{DC}$. The figure tells us that $\overrightarrow{BC}$ and $\overrightarrow{DA}$ are oppositely directed [and that $\overrightarrow{BA}$ and $\overrightarrow{DC}$ are oppositely directed]. So, we can use Theorem 5-14.

Solution.

(1)	$\overrightarrow{BC} \parallel \overrightarrow{DA}$ and $\overrightarrow{BA} \parallel \overrightarrow{DC}$	[Hypothesis]
(2)	$\overrightarrow{BC}$ and $\overrightarrow{DA}$ are oppositely directed and $\overrightarrow{BA}$ and $\overrightarrow{DC}$ are oppositely directed	[(1); figure]
(3)	______________________	[theorem]
(4)	$\angle B \cong \angle D$	[(2) and (3)]
(5)	$\angle A \cong \angle C$	[Steps like (1) - (3)]

* * *

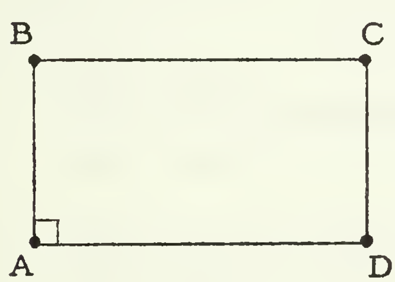
J. 1.



Hypothesis: $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$, $\overleftrightarrow{AD} \perp \overleftrightarrow{DC}$, $\overleftrightarrow{DC} \perp \overleftrightarrow{BC}$

Conclusion: $\overleftrightarrow{BA} \perp \overleftrightarrow{DA}$

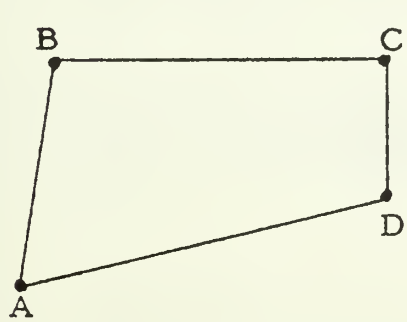
2.



Hypothesis: $\overline{AB} \parallel \overline{DC}$, $\overline{BC} \parallel \overline{AD}$, $\overline{BA} \perp \overline{AD}$

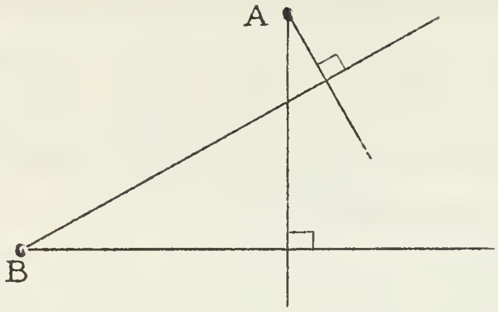
Conclusion: $\angle B$, $\angle C$, and $\angle D$ are right angles

3.



Hypothesis: [figure as shown]

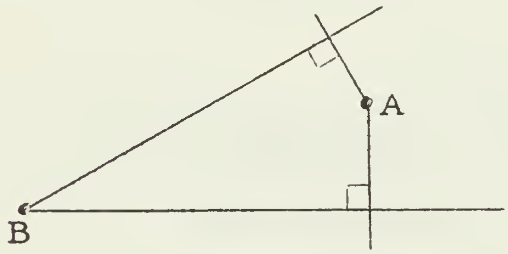
Conclusion: $m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = 360$

4. 

<u>Hypothesis:</u> the sides of $\angle A$ and $\angle B$ are perpendicular as shown

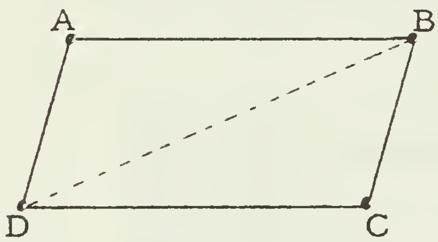
<u>Conclusion:</u> $\angle A \cong \angle B$

[<u>Hint.</u> Consider the sum of the measures of the three angles of each of the two triangles shown in the figure.]

5. 

<u>Hypothesis:</u> the sides of $\angle A$ and $\angle B$ are perpendicular as shown

<u>Conclusion:</u> $\angle A$ and $\angle B$ are supplementary

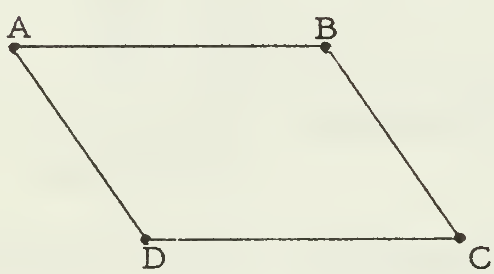
6. 

<u>Hypothesis:</u> $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$

<u>Conclusion:</u> $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$

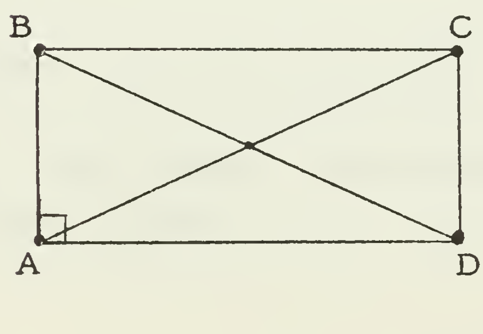
7. [Interchange the Hypothesis and Conclusion of Exercise 6.]

8. If $ABD \longleftrightarrow CBD$ is a congruence, does it follow that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$?

9. 

<u>Hypothesis:</u> $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, $AB = DC$

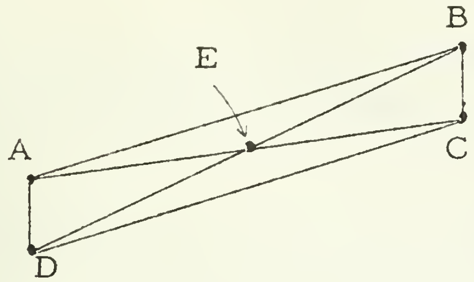
<u>Conclusion:</u> $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$, $AD = BC$

10. 

<u>Hypothesis:</u> [as in Exercise 2]

<u>Conclusion:</u> $\overline{AC} \cong \overline{BD}$

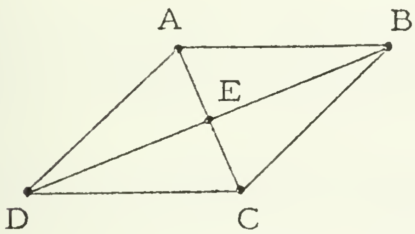
11.



Hypothesis: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$, $\overleftrightarrow{BD} \cap \overleftrightarrow{AC} = \{E\}$

Conclusion: $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ bisect each other

12.



Hypothesis: [as in Exercise 11] also: $\overleftrightarrow{AB} \cong \overleftrightarrow{AD}$

Conclusion: $\overleftrightarrow{AC} \perp \overleftrightarrow{BD}$ at E

[Supplementary exercises are on page 6-426.]

SUMMARY OF SECTION 6.05

Notation and Terminology

alternate exterior angles	[6-143]	interior angles	[6-142]
alternate interior angles	[6-142]	oppositely directed rays	[6-153]
consecutive exterior angles	[6-143]	parallel segments, half-lines, rays	[6-152]
consecutive interior angles	[6-143]	similarly directed rays	[6-153]
corresponding angles	[6-143]	transversal	[6-142]
exterior angles	[6-142]		

Theorems

5-1. $\forall_m \forall_\ell \forall_{P \notin \ell}$ m is the parallel to ℓ through P if and only if $P \in m$ and $m \cap \ell = \emptyset$

5-2. If two lines are cut by a transversal in such a way that some two alternate interior angles are congruent then the two lines are parallel.

5-3. If two parallel lines are cut by a transversal then each two alternate interior angles are congruent.

5-4. If two lines are cut by a transversal in such a way that some two consecutive interior angles are supplementary then the two lines are parallel.

5-5. If two parallel lines are cut by a transversal then each two consecutive interior angles are supplementary.

5-6. If two lines are cut by a transversal in such a way that some two corresponding angles are congruent then the two lines are parallel.

5-7. If two parallel lines are cut by a transversal then each two corresponding angles are congruent.

5-8. Two lines which are perpendicular to a third line are parallel.

5-9. If one of two parallel lines is perpendicular to a third line then so is the other.

5-10. The measure of an exterior angle of a triangle is the sum of the measures of the two opposite angles of the triangle.

5-11. The sum of the measures of the angles of a triangle is 180.

5-12. The measure of the shorter leg of a 30-60-90 triangle is half the measure of the hypotenuse.

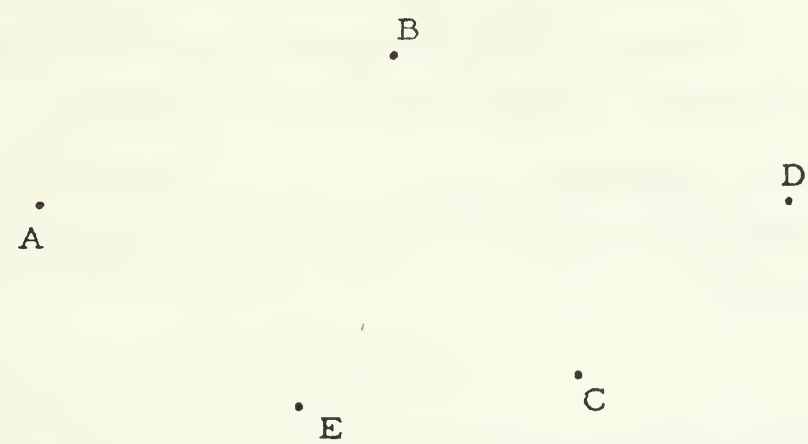
5-13. If the sides of two angles can be matched in such a way that corresponding sides are similarly directed then the angles are congruent.

5-14. If the sides of two angles can be matched in such a way that corresponding sides are oppositely directed then the angles are congruent.

5-15. If the sides of two angles can be matched in such a way that two corresponding sides are similarly directed and the other two corresponding sides are oppositely directed then the angles are supplementary.

EXPLORATION EXERCISES

A. Here is a map showing the locations of several points of interest to an idle fellow. He started from A and walked directly to one of the



other points. From there he walked directly to another. Etc. Having visited each of the points just once he returned directly to A.

1. Mark [in pencil] one possible path he could have followed.
2. Mark [in ink] another possible path.
3. How many such paths are there?

B. Each of the paths from point A back to point A in the preceding part is called a closed polygonal path. And, a closed polygonal path which doesn't cross itself is called a simple closed polygonal path. Can the idle fellow follow a simple closed polygonal path on his tour?

* * *

Each of the closed polygonal paths of Part A is the union of five segments such that

(1) each end point is an end point of just two segments.

A simple closed polygonal path [Part B] has the additional property that

(2) no two segments intersect except at an end point.

These properties are characteristic of all simple closed polygonal paths.

* * *

C. 1. Is a triangle a simple closed polygonal path from one vertex back to itself?

2. Mark four points A, B, C, and D such that no three are collinear. Is there a closed polygonal path from A back to A which is not simple? Is there one which is simple?

3. Mark four noncollinear points A, B, C, and D such that three are collinear. Is there a simple closed polygonal path from A back to A?

* * *

In our work in this section we shall be interested in simple closed polygonal paths which satisfy a third condition:

(3) no two segments with a common end point are collinear

A set which is the union of segments satisfying the conditions:

(1) each end point is an end point of just two segments,

(2) no two segments intersect except at an end point,

(3) no two segments with a common end point are collinear,

is called a polygon. The segments are called the sides of the polygon and their end points are called the vertices of the polygon. Do you see that a polygon has the same number of sides as vertices? Adjacent sides of a polygon are sides that share a vertex. What is an angle of a polygon?

* * *

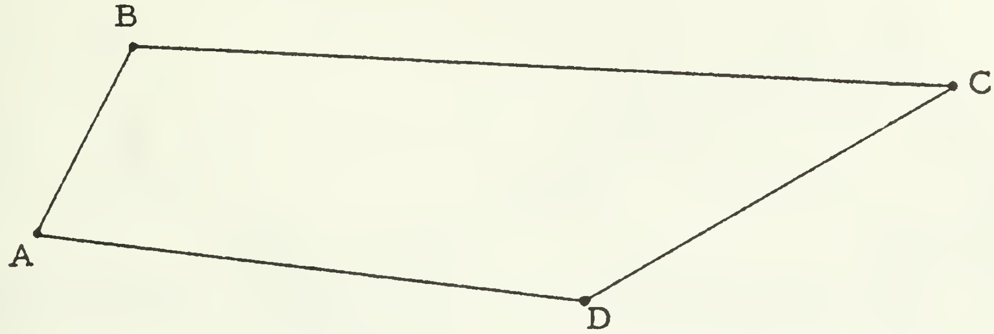
D. 1. Is the simple closed polygonal path in Exercise 3 of Part C a polygon?

2. What is a common name for a polygon of three sides?

3. Draw a four-sided polygon.

4. Draw a polygon which has six vertices.

E. Two vertices of a polygon which are end points of a side are called adjacent vertices. You can name a polygon by listing its vertices starting with one of them and continuing with names of adjacent vertices. So, for example, here is a polygon ABCD:



This polygon can also be called 'ADCB' or 'CDAB', or 'BADC', etc. But, it shouldn't be called 'BDAC', for example. [For the four points shown, is there a polygon called 'BDAC'?]

1. Draw a polygon UICSM. Draw another polygon SIAM.

2. Draw a polygon UICSM such that it is impossible to draw a polygon SIAM.

3. Draw a polygon ABC.

   (a) Show, by shading, the set of all points D such that ABCD is a polygon.

   (b) Show, by shading, the set of all points D such that A, B, C, and D are the vertices of a polygon.

F. A segment whose end points are nonadjacent vertices of a polygon is called a diagonal of the polygon.

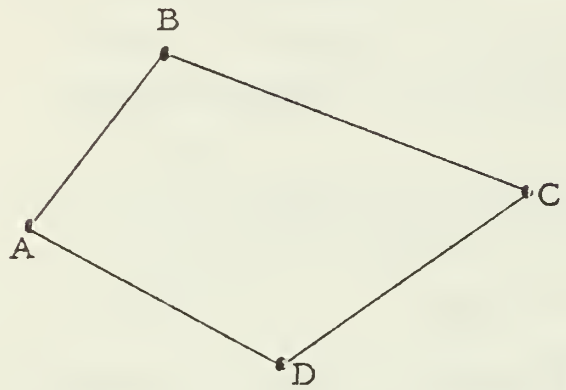
1. Draw a polygon ABCDE. Draw its diagonals. Name them. How many are there?

2. How many diagonals does a polygon of four sides have? Do the diagonals of a four-sided polygon intersect?

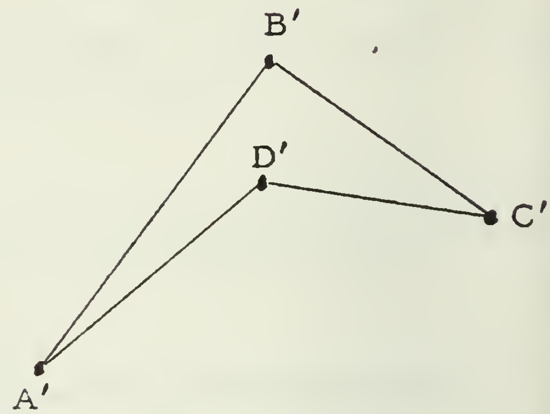
3. Complete the following table:

number of sides	3	4	5	6	7	8	20	100	10000
number of diagonals									

G. Here are two four-sided polygons. One is convex and the other is nonconvex.



convex polygon



nonconvex polygon

1. Draw a convex polygon DEFGH.

2. Draw a nonconvex polygon MNOPQR.

*

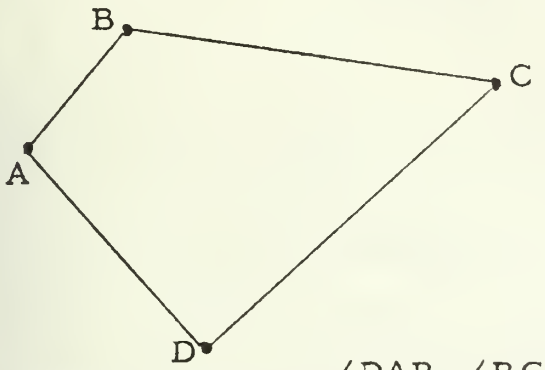
Ted has drawn a polygon ABCDE . . . . He wants to know if it is convex or nonconvex. Bill tells him over the telephone to imagine that the polygon has been drawn on a flat wooden surface and that nails have been driven part way into the wood at the vertices. Think of a noose placed loosely around the nails. The polygon is convex if and only if the noose touches all the nails when it is tightened.

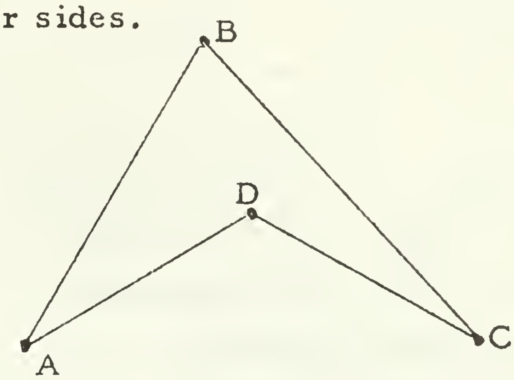
*

3. Draw the diagonals of a four-sided convex polygon and the diagonals of a four-sided nonconvex polygon. What difference do you notice?

4. Draw a polygon ABC.
   (a) Show, by shading, the set of all points D such that ABCD is a convex polygon.
   (b) Show, by shading, the set of all points D such that A, B, C, and D are the vertices of a convex polygon.

6.06 <u>Quadrilaterals.</u> --A four-sided polygon is called <u>a</u> <u>quadrilateral.</u> A quadrilateral has four angles and four sides.





<u>opposite</u> <u>angles</u> $\begin{cases} \angle DAB, \angle BCD \\ \angle ABC, \angle CDA \end{cases}$

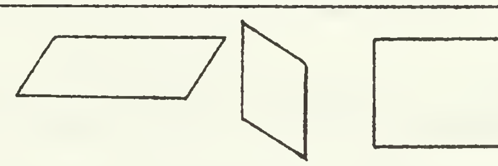
<u>opposite</u> <u>sides</u> $\begin{cases} \overline{AB}, \overline{CD} \\ \overline{BC}, \overline{DA} \end{cases}$

Two angles of a quadrilateral are <u>adjacent</u> if their vertices are adjacent; two <u>sides</u> are <u>adjacent</u> if they share a vertex.

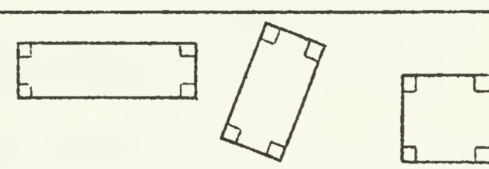
TYPES OF QUADRILATERALS

In earlier grades you learned about various types of quadrilaterals. We give below definitions and illustrations.

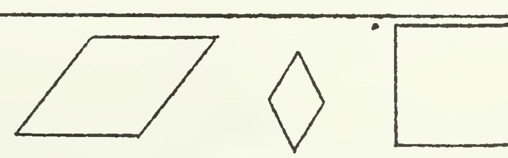
A quadrilateral is a <u>parallelogram</u> if and only if its opposite sides are parallel.



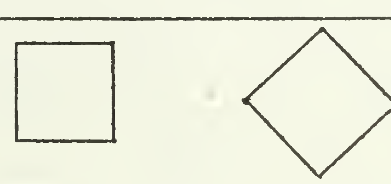
A quadrilateral is a <u>rectangle</u> if and only if its angles are right angles.



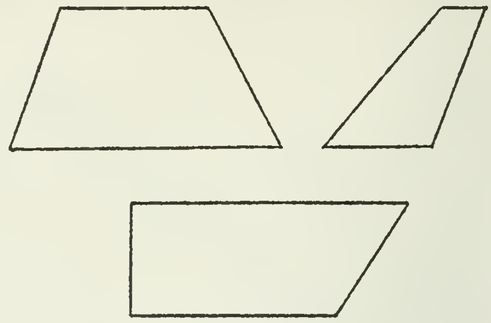
A quadrilateral is a <u>rhombus</u> if and only if all its sides are congruent.



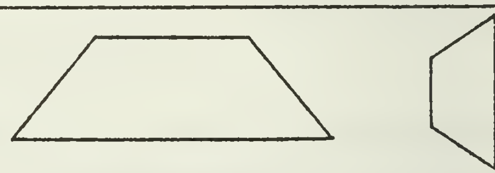
A quadrilateral is a <u>square</u> if and only if it is a rectangle with two adjacent sides congruent.



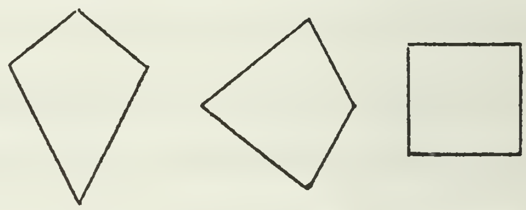
A quadrilateral is a *trapezoid* if and only if just two of its sides are parallel. [The parallel sides are the *bases*, the nonparallel sides are the *legs*, and the pairs of angles which share a base are pairs of *base angles*.]



A quadrilateral is an *isosceles trapezoid* if and only if it is a trapezoid whose legs are congruent.



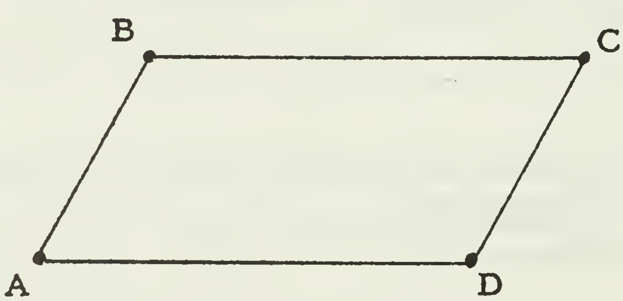
A quadrilateral is a *kite* if and only if it is convex and two adjacent sides are congruent and the other two sides are congruent.



Example 1. Prove: The opposite sides of a parallelogram are congruent.

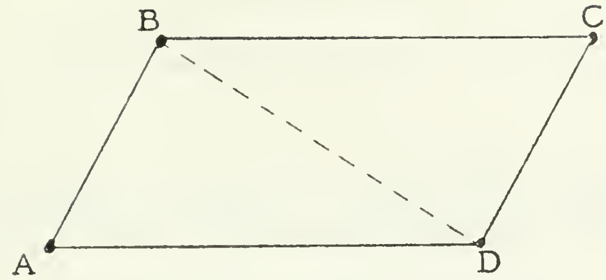
Plan. Let us first state a conditional sentence which is an instance of this theorem:

if ABCD is a parallelogram then $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{CB}$



We can show that $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{CB}$ by showing that they are corresponding parts of congruent triangles. This suggests drawing $\overline{BD}$ or $\overline{AC}$ in the figure. A matching which appears to be a congruence is ABD $\longleftrightarrow$ CDB. Since the definition of a parallelogram tells us that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$, we can use the theorems about parallel lines and angles to show $\angle ABD \cong \angle CDB$ and $\angle CBD \cong \angle ADB$. Then, use a.s.a.

Solution.



(1)	ABCD is a parallelogram	[assumption] *
(2)	$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	[(1); def. of parallelogram]
(3)	$\angle ABD$ and $\angle CDB$ are alternate interior angles	[figure]
(4)	____________________	[theorem]
(5)	$\angle ABD \cong \angle CDB$	[(3) and (4)]
(6)	$\angle BDA \cong \angle DBC$	[Steps like (1) - (5)]
(7)	$\overleftrightarrow{BD} \cong \overleftrightarrow{DB}$	[Identity; def. of cong. segs.]
(8)	A, B, D and C, D, B are vertices of triangles	[figure]
(9)	____________________	[theorem]
(10)	$ABD \longleftrightarrow CDB$ is a congruence	[(8), (5), (7), (6), and (9)]
(11)	$\overleftrightarrow{AB} \cong \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \cong \overleftrightarrow{CB}$	[(10); def. of congruence]
(12)	if ABCD is a parallelogram then $\overleftrightarrow{AB} \cong \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \cong \overleftrightarrow{CB}$	[(11); * (1)]
(13)	$\boxed{\text{The opposite sides of a parallelogram are congruent.}}$	[(1) - (12)]

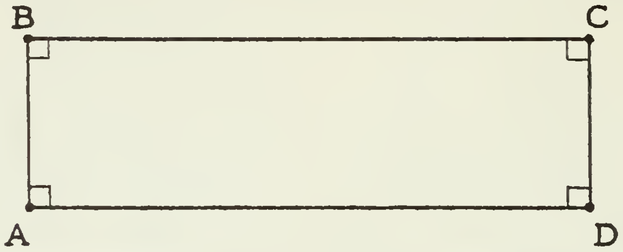
* * *

The example illustrates how you use definitions of the special types of quadrilaterals in proofs [Step (2)]. On the next page is another example of the use of definitions in proofs. Write a plan for the proof of the theorem 'Each rectangle is a parallelogram'. It may not be the same as the one on the next page but this simply shows that there is often more than one way to prove a theorem.

* * *

Example 2. Prove that each rectangle is a parallelogram.

Plan.



We can prove that rectangle ABCD is a parallelogram by showing that opposite sides are parallel. Since the definition of a rectangle tells us that all the angles of ABCD are right angles, we can use the theorem about two lines perpendicular to a third line.

Solution.

(1)	ABCD is a rectangle	[assumption] *
(2)	$\angle A$ is a right angle	[(1); def. of rectangle]
(3)	$\overleftrightarrow{DA} \perp \overleftrightarrow{AB}$	[(2); def. of perpendicular]
(4)	$\overleftrightarrow{CB} \perp \overleftrightarrow{AB}$	[Steps like (2) - (3)]
(5)	______________________	[theorem]
(6)	$\overleftrightarrow{DA} \parallel \overleftrightarrow{CB}$	[(3), (4), and (5)]
(7)	$\overleftrightarrow{BA} \parallel \overleftrightarrow{CD}$	[Steps like (2) - (6)]
(8)	ABCD is a parallelogram	[(6) and (7); def. of parallelogram]
(9)	if ABCD is a rectangle then ABCD is a parallelogram	[(8); * (1)]
(10)	**Each rectangle is a parallelogram.**	

The examples are proofs of two theorems about quadrilaterals. Before reading further in the textbook, we suggest that you investigate other properties of quadrilaterals, express the properties as theorems, and then prove the theorems. Some of the exercises in Part J starting on page 6-155 should give you a start on this project.

After your class has made a list of some such theorems, you shoul work on the following exercises, using the class theorems whenever possible. The exercises may suggest other theorems to add to the list.

EXERCISES

1. Using compass and straight-edge only, draw each of the figures described below.

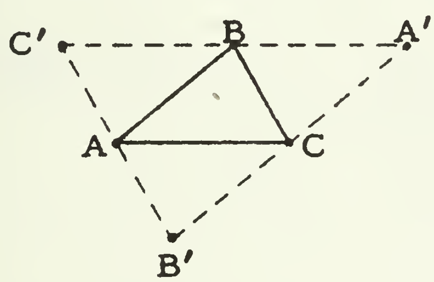
   (a) A parallelogram ABCD for which $\overline{AB}$ is congruent to the given segment $\overline{EF}$, $\overline{BC} \cong \overline{GH}$, and $\angle D \cong \angle J$.

   (b) A square one of whose sides is congruent to $\overline{AB}$.

   (c) A quadrilateral whose diagonals bisect each other and are congruent to $\overline{CD}$.

2. Given a parallelogram ABCD. If $m(\angle A) = 2 \cdot m(\angle B)$, give the number of degrees in each of the four angles.

3. (a)



<u>Given:</u> $\overleftrightarrow{A'B'}$, $\overleftrightarrow{B'C'}$, and $\overleftrightarrow{C'A'}$ are lines through the vertices of $\triangle ABC$ and parallel to the sides

<u>Find:</u> the measures of $\angle A'$, $\angle B'$, and $\angle C'$

   (b) Show that each of the following matchings is a congruence:
   (1) $ABC \longleftrightarrow A'CB$ (2) $ABC \longleftrightarrow BAC'$ (3) $ABC \longleftrightarrow CB'A$

   (c) Show that A, B, and C are the midpoints of the sides of $\triangle B'C'A'$.

   (d) Show that the altitude from B of $\triangle ABC$ is contained in the perpendicular bisector of side $\overline{C'A'}$ of $\triangle B'C'A'$.

   (e) Prove:

$\boxed{\text{The altitudes of a triangle are concurrent.}}$

4. $\angle A$ is an angle of 60° and $\overrightarrow{AP}$ is its bisector. The points B and C are points in the sides of $\angle A$ such that $\overline{PB} \parallel \overline{AC}$ and $\overline{PC} \parallel \overline{AB}$.

   (a) Show that ABPC is a rhombus.

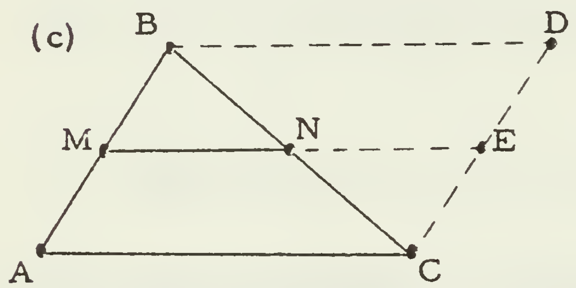
   (b) Show that one of the diagonals of ABPC is congruent to $\overline{AB}$.

5. Given a parallelogram ABCD. Show that the vertices B and D are equidistant from $\overleftrightarrow{AC}$.

6. (a) Show that the line which bisects one side of a parallelogram and is parallel to an adjacent side also bisects the opposite side.

   (b) The segment joining the midpoints of two opposite sides of a parallelogram is parallel and congruent to the other sides.

   (c)



<u>Hypothesis:</u> ABDC is a parallelogram, M is the midpoint of $\overline{AB}$, $\overline{MN} \parallel \overline{AC}$

<u>Conclusion:</u> N is the midpoint of $\overline{BC}$

[<u>Hint.</u> Use Exercise 6 (a).]

   (d) [Refer to the figure for part (c).] Given that ABDC is a parallelogram, M is the midpoint of $\overline{AB}$, and N is the midpoint of $\overline{BC}$. Show that $\overline{MN} \parallel \overline{AC}$ and that $MN = \frac{1}{2} \cdot AC$.

*

The work you did in parts (c) and (d) leads to the following important theorems:

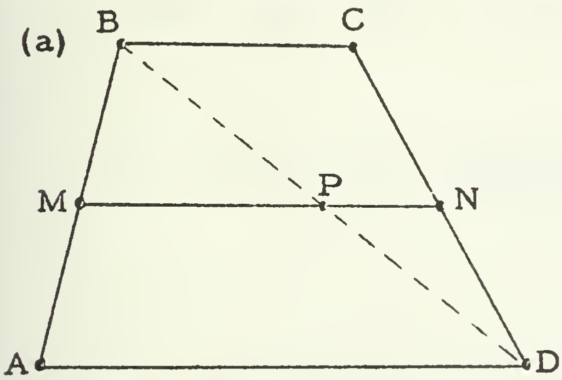
The line which bisects one side of a triangle and is parallel to a second side bisects the third.

The segment joining the midpoints of two sides of a triangle is parallel to the third side, and its measure is one half the measure of the third side.

*

7. The perimeter of ΔABC is 30. [That is, $AB + BC + CA = 30$.] If M, N, and P are the midpoints of the three sides of ΔABC, what is the perimeter of ΔMNP?

8. (a)



<u>Hypothesis:</u> ABCD is a trapezoid with legs $\overline{BA}$ and $\overline{CD}$, M is the midpoint of $\overline{BA}$, N is the midpoint of $\overline{CD}$

<u>Conclusion:</u> the line through M and parallel to $\overline{AD}$ is $\overleftrightarrow{MN}$

[<u>Hint.</u> Suppose the line through M and parallel to $\overline{AD}$ intersects $\overline{BD}$ in P. Then, consider ΔABD and ΔDCB and use the first of the two theorems displayed on page 6-168.]

(b) The segment joining the midpoints of the legs of a trapezoid is called <u>the median of the trapezoid</u>. In part (a) you proved:

The median of a trapezoid is parallel to the bases.

Now, prove:

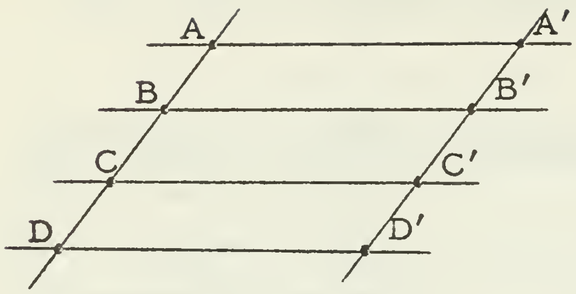
The measure of the median of a trapezoid is the average of the measures of the bases.

9. (a) Draw a quadrilateral. [It need not be convex.] Locate the midpoints of the four sides. Draw the segments which join the midpoints of adjacent sides. What kind of quadrilateral has these segments as sides? Prove your conjecture.

(b) Try this for the special quadrilaterals.

10. Quadrilateral $ABA'B'$ is a parallelogram. Suppose that C and C' are two points such that $B \in \overline{AC}$ and $B' \in \overline{A'C'}$ and $ACA'C'$ is a parallelogram. Show that $BCB'C'$ is a parallelogram.

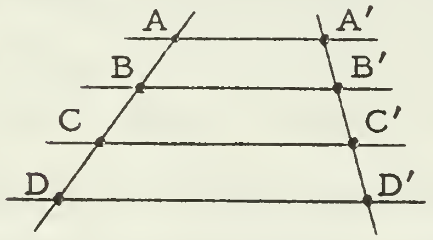
11. (a)



Hypothesis: $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$, and $\overleftrightarrow{DD'}$ are parallel lines, $\overleftrightarrow{AD} \parallel \overleftrightarrow{A'D'}$, $AB = BC = CD$

Conclusion: $A'B' = B'C' = C'D'$

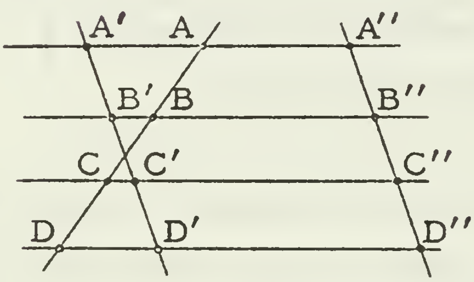
(b)



Hypothesis: As in part (a), except that $\overleftrightarrow{AD}$ is not parallel to $\overleftrightarrow{A'C'}$

Conclusion: As in part (a).

(c)



Hypothesis: As in part (b).

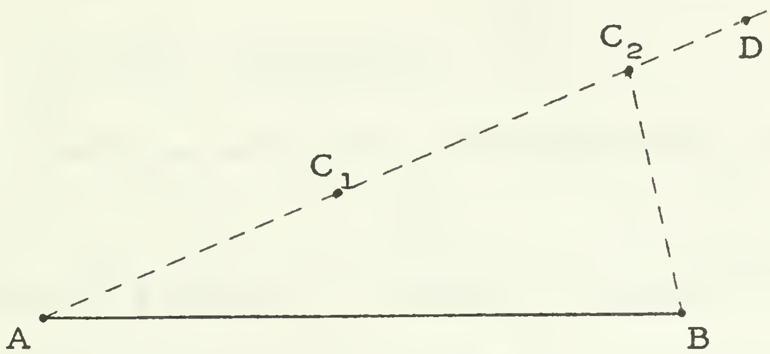
Conclusion: As in part (b).

*

Exercise 11 gives us the following theorem:

> If parallel lines cut off congruent segments on one transversal, they cut off congruent segments on every transversal.

This theorem is useful in locating the midpoint of a segment. For example, suppose you are given the segment $\overline{AB}$ and you want to use just your compass and straight-edge to locate its midpoint. Draw a half-line $\overrightarrow{AD}$ in one side of $\overleftrightarrow{AB}$. Locate points C_1 and C_2 on $\overrightarrow{AD}$ such that



$AC_1 = C_1C_2$. Draw $\overline{C_2B}$. Then, draw the line through C_1 which is parallel to $\overline{C_2B}$. This line intersects $\overline{AB}$ in a point M.

Now, consider the lines $\overleftrightarrow{C_1M}$, $\overleftrightarrow{C_2B}$, and the line through A which is parallel to $\overleftrightarrow{C_2B}$. These three parallel lines cut off congruent segments, $\overline{AC_1}$ and $\overline{C_1C_2}$, on the transversal $\overleftrightarrow{AD}$. So, they must cut off congruent segments on the transversal $\overleftrightarrow{AB}$. Hence, AM = MB, and M is the midpoint of $\overline{AB}$.

*

12. (a) Use the method discussed above to trisect a segment.

    (b) Use it to divide a segment into 7 congruent segments.

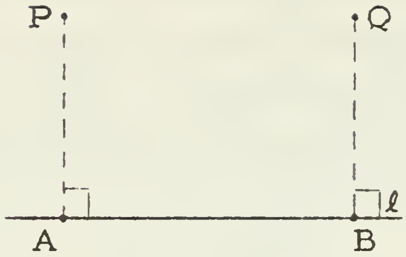
13. Quadrilateral ABCD is a parallelogram. Let P and Q be the trisection points of $\overline{AC}$, and let M and N be the midpoints of $\overline{AB}$ and $\overline{CD}$, respectively. Draw the quadrilateral MPNQ. Is this a parallelogram? Prove it.

14. (a) The measure of the hypotenuse $\overline{AB}$ of right triangle ΔABC is 10. What is the measure of the median from C?

    (b) State and prove the theorem suggested by part (a).

    (c) Compare the theorem of (b) with the theorem you proved in Exercise 6(a) on page 6-150. State another theorem which combines these two theorems.

15. (a)



<u>Hypothesis:</u> P and Q are two points on the same side of ℓ, P and Q are equidistant from ℓ

<u>Conclusion:</u> $\overleftrightarrow{PQ} \parallel \ell$

(b) Interchange the Hypothesis and Conclusion of (a).

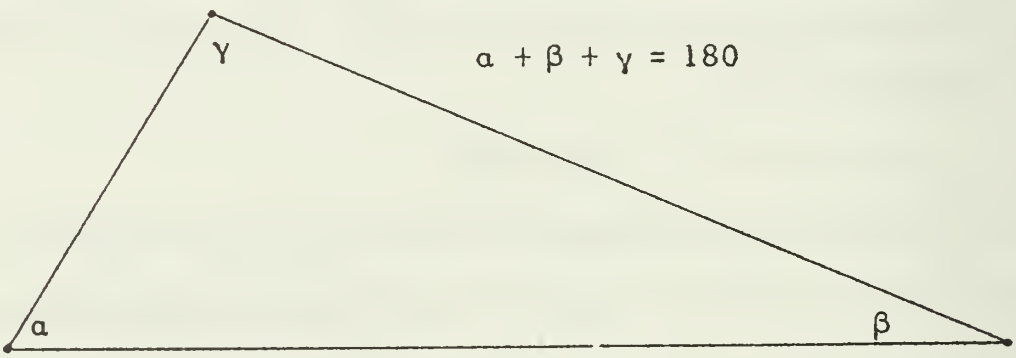
*

Your work in Exercise 15 justifies the following theorem:

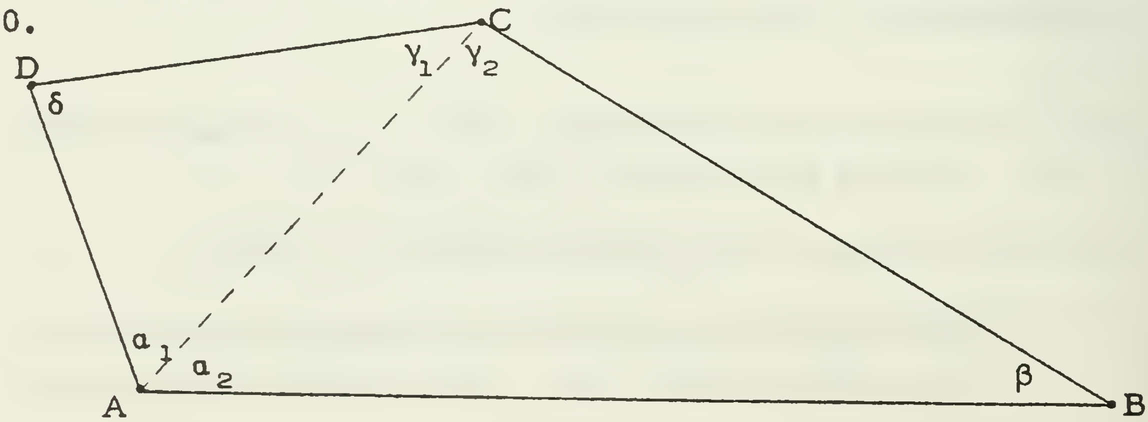
Parallel lines are everywhere equidistant.

ANGLES OF A CONVEX POLYGON

You have already proved that the sum of the measures of the angles of any triangle is 180.



It is easy to see that the sum of the measures of the four angles of a rectangle, parallelogram, or trapezoid is 360. [Justify this.] This suggests that the sum of the measures of the angles of any quadrilateral is 360.



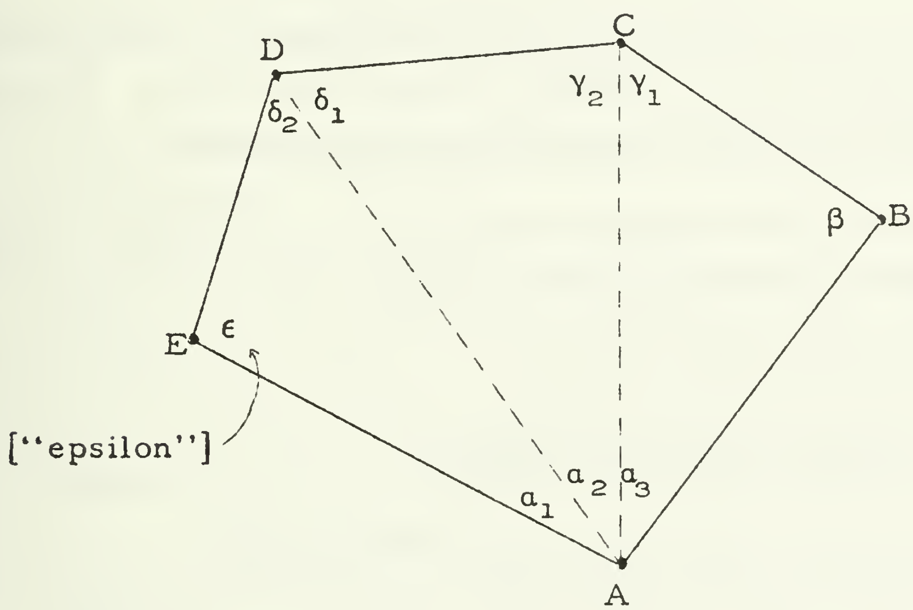
Since $\alpha_1 + \gamma_1 + \delta = 180$ and $\alpha_2 + \beta + \gamma_2 = 180$, it follows that

$$(\alpha_1 + \alpha_2) + \beta + (\gamma_1 + \gamma_2) + \delta = 360.$$

But, since C is interior to $\angle DAB$, $\alpha_1 + \alpha_2$ is the measure of $\angle A$. Similarly, $\gamma_1 + \gamma_2$ is the measure of $\angle C$. So, the sum of the measures of the angles of quadrilateral ABCD is 360.

Can you draw a quadrilateral to which this argument would not apply, say, a quadrilateral for which C is not interior to $\angle DAB$?

Now, consider the sum of the measures of the angles of a convex polygon of five sides. Here is a picture of the convex pentagon ABCDE:



Prove that the sum of the measures of the angles of the pentagon is 540.

EXERCISES

A. 1. Find the sum of the measures of the angles of a convex hexagon.

2. Find the sum of the measures of the angles of a convex dodecagon.

3. Find the sum of the measures of the angles of a convex polygon of 1002 sides.

4. Find the sum of the measures of the angles of convex polygon of n sides.

B. A <u>regular polygon</u> is a convex polygon all of whose angles are congruent and all of whose sides are congruent.

1. Find the number of degrees in each angle of regular polygon of 5 sides.

2. Complete the following table for the number of degrees in each angle of a regular polygon.

No. of sides	3	4	5	6	10	1002	n
No. of degrees							

C. 1. You know what an exterior angle of a triangle is. This idea can be extended to convex polygons of more than 3 sides. State a definition of an <u>exterior angle of a convex polygon</u>.

2. Find the sum of the measures of three exterior angles, one at each vertex, of an equilateral triangle.

3. Repeat Exercise 2 for a square.

4. Repeat Exercise 2 for any triangle.

5. Repeat Exercise 2 for any convex quadrilateral.

6. Find the sum of the measures of n exterior angles, one at each vertex, of a convex polygon of n sides.

7. Find the number of degrees in an exterior angle of a regular polygon of 10 sides.

8. Repeat Exercise 7 for a regular polygon of n sides.

D. An <u>equiangular polygon</u> is a polygon all of whose angles are congruent. An <u>equilateral polygon</u> is a polygon all of whose sides are congruent.

1. What is a common name for an equilateral quadrilateral?

2. What is a common name for an equiangular quadrilateral?

3. What is a common name for a regular quadrilateral?

4. Guess which of the following are theorems.

   (a) Each equiangular triangle is equilateral.
   (b) Each equilateral triangle is equiangular.
   (c) Each equiangular quadrilateral is equilateral.
   (d) Each equilateral quadrilateral is equiangular.
   (e) Each equilateral triangle is regular.
   (f) Each equiangular triangle is regular.
   (g) Each equiangular pentagon is regular.
   (h) Each equilateral pentagon is regular.
   (i) Each equilateral convex pentagon is regular.
   (j) Each equiangular hexagon is regular.
   (k) Each equiangular polygon is convex.

<u>E.</u> 1. What is the measure of an exterior angle of a regular polygon of 12 sides?

2. What is the sum of the measures of the angles of a [convex] polygon of 5 sides?

3. How many sides has a regular polygon if one of its exterior angles is an angle of 45° ?

4. An exterior angle of a regular <u>decagon</u> is an angle of _______ degrees.

5. Find the sum of the measures of the angles of a polygon of 12 sides .

6. Is there a regular polygon each of whose exterior angles is an angle of 70° ?

7. What is the measure of an angle of an equiangular polygon of 6 sides?

8. What is the number of sides of a regular polygon the sum of the measures of whose angles is 720?

9. The sum of the measures of the angles of a polygon of ______ sides is the sum of the measures of its exterior angles.

SUMMARY OF SECTION 6.06

Notation and Terminology

Theorems

6-1. The opposite sides of a parallelogram are congruent.

6-2. Each rectangle is a parallelogram.

6-3. Adjacent angles of a parallelogram are supplementary.

6-4. The opposite angles of a parallelogram are congruent.

6-5. The diagonals of a parallelogram bisect each other.

6-6. If the opposite sides of a quadrilateral are congruent, the quadrilateral is a parallelogram.

6-7. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.

6-8. If two sides of a quadrilateral are parallel and congruent, the quadrilateral is a parallelogram.

6-9. If each two adjacent angles of a quadrilateral are supplementary, the quadrilateral is a parallelogram.

6-10. If each two opposite angles of a quadrilateral are congruent, the quadrilateral is a parallelogram.

6-11. The diagonals of a rectangle are congruent.

6-12. If the diagonals of a parallelogram are congruent, the parallelogram is a rectangle.

6-13. A rhombus is a parallelogram.

6-14. If two adjacent sides of a parallelogram are congruent, the parallelogram is a rhombus.

6-15. The diagonals of a kite are perpendicular to each other.

6-16. The diagonals of a rhombus are perpendicular bisectors of each other.

6-17. If the diagonals of a quadrilateral are perpendicular bisectors of each other, the quadrilateral is a rhombus.

6-18. The diagonals of a rhombus are contained in the bisectors of its angles.

6-19. If the diagonals of a quadrilateral are contained in the bisectors of its angles, the quadrilateral is a rhombus.

6-20. If a pair of base angles of a trapezoid are congruent, the trapezoid is isosceles.

6-21. Each pair of base angles of an isosceles trapezoid are congruent.

6-22. The altitudes of a triangle are concurrent.

6-23. The line which bisects one side of a triangle and is parallel to a second side bisects the third.

6-24. The segment joining the midpoints of two sides of a triangle is parallel to the third side, and its measure is one half the measure of the third side.

6-25. The median of a trapezoid is parallel to the base.

6-26. The measure of the median of a trapezoid is the average of the measures of the bases.

6-27. If parallel lines cut off congruent segments on one transversal, they cut off congruent segments on every transversal.

6-28. A triangle is a right triangle if and only if the measure of the median to one of its sides if half the measure of that side.

6-29. Parallel lines are everywhere equidistant.

6-30. The sum of the measures of the angles of a convex polygon of n sides is $(n - 2)180$.

6-31. The degree measure of an angle of a regular polygon of n sides is $\left(\frac{n - 2}{n}\right) 180$.

6-32. The sum of the measures of n exterior angles, one at each vertex, of a convex polygon of n sides is 360.

6-33. The degree-measure of an exterior angle of a regular polygon of n sides is $\frac{360}{n}$.

[Supplementary exercises are on page 6-428.]

EXPLORATION EXERCISES

Consider the sentence:

(*) ABCD is a rhombus

This sentence is implied by [or follows from] the sentence:

(1) ABCD is a square

because every square is a rhombus. But,(*) does not imply (1) because some rhombuses are not squares.

Complete the following table:

	Is implied by (*)	Implies (*)
(1) ABCD is a square	No	Yes
(2) ABCD is a parallelogram	___	___
(3) ABCD is a rectangle	___	___
(4) the diagonals of ABCD are perpendicular	___	___
(5) the diagonals of ABCD are perpendicular bisectors of each other	___	___
(6) the sides of ABCD are congruent	___	___
(7) the diagonals of ABCD are congruent	___	___
(8) the diagonals of ABCD bisect each other	___	___
(9) ABCD is a parallelogram with two adjacent sides congruent	___	___
(10) ABCD is a kite	___	___
(11) the diagonals of ABCD are contained in the bisectors of its angles	___	___
(12) ABCD is equilateral	___	___
(13) ABCD is equiangular	___	___
(14) ABCD is a regular polygon	___	___

NECESSARY AND SUFFICIENT CONDITIONS

A sentence which implies a given sentence is sometimes called <u>a sufficient condition</u> for that sentence. Which of the sentences given on page 6-179 are sufficient conditions for (*)?

[The sentence (*) is a sufficient condition for some of the sentences on page 6-179 because it implies them. For example, (*) is a sufficient condition for (2). For what others is (*) a sufficient condition?]

To claim that a sentence p is a sufficient condition for a given sentence q amounts to claiming that the conditional sentence

if p then q

is a theorem.

A sentence which is implied by a given sentence -- that is, one which is a necessary consequence of it -- is called <u>a necessary condition</u> for that sentence. Which of the sentences on page 6-179 are necessary conditions for (*)?

[For which of them is (*) a necessary condition?]

To claim that the conditional sentence

if p then q

is a theorem amounts to claiming that q is a necessary condition for p.

$$\left.\begin{array}{l}\text{if p then q}\\ \text{q if p}\\ \text{p only if q}\end{array}\right\}\text{ is a theorem }\left<\begin{array}{l}\text{p is a sufficient condition for q}\\ \text{q is a necessary condition for p}\end{array}\right.$$

So, for example, since we know [from an earlier unit] that

'if a = 0 then ab = 0' is a theorem,

we can also claim that

(a) 'a = 0' is a sufficient condition for 'ab = 0',

(b) 'ab = 0' is a necessary condition for 'a = 0',

(c) 'ab = 0 if a = 0' is a theorem,

and that (d) 'a = 0 only if ab = 0' is a theorem.

EXERCISES

A. Give four translations, as above, for each of the following:

1. 'if ΔABC is equilateral then ΔABC is isosceles' is a theorem.
2. 'if ABCD is a rectangle then the diagonals of ABCD are congruent' is a theorem.
3. 'if ABCD is equilateral then ABCD is a rhombus' is a theorem.
4. 'if ABCD is a rhombus then ABCD is equilateral' is a theorem.

* * *

Notice that one of your answers in Exercise 4 is:

'ABCD is equilateral' is a necessary condition for 'ABCD is a rhombus'.

Also, notice that one of your answers in Exercise 3 is:

'ABCD is equilateral' is a sufficient condition for 'ABCD is a rhombus'.

So, putting Exercises 4 and 3 together -- that is, saying that

'if ABCD is a rhombus then ABCD is equilateral'
and
'if ABCD is equilateral then ABCD is a rhombus'

are theorems -- we can say that

necessary
(I) 'ABCD is equilateral' is a and condition for 'ABCD is a rhombus'.
sufficient

Also, we can say that

necessary
(II) 'ABCD is a rhombus' is a and condition for 'ABCD is equilateral',
sufficient

and that

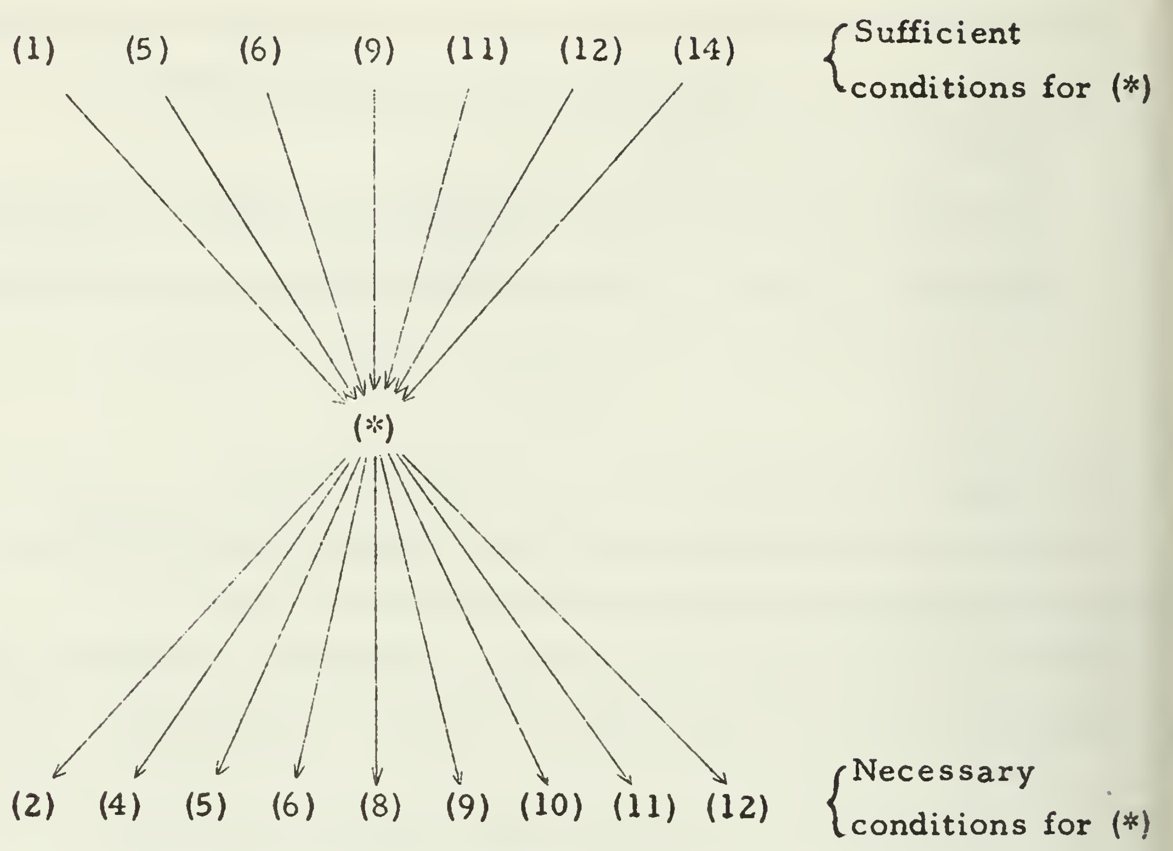
if
(III) 'ABCD is equilateral and ABCD is a rhombus' is a theorem,
only if

and that

if
(IV) 'ABCD is a rhombus and ABCD is equilateral' is a theorem.
only if

* * *

<u>B.</u> The correct answers to the exercise on page 6-179 are summarized in the following diagram:



Notice that some of the sentences are both necessary and sufficient conditions for (*). The fact that sentence (12) is one such sentence is expressed by statements (I), (II), (III), and (IV) at the bottom of page 6-181.

Pick another of the sentences which is both a necessary and sufficient condition for (*). Express this by making four statements like (I), (II), (III), and (IV).

* * *

	if				necessary	
p	and	q	is a theorem ---	p is a	and	condition for q
	only if				sufficient	

* * *

C. In each of the following exercises you are given a figure, a sentence (*), and some other sentences each of which may be a necessary or sufficient condition for (*). Indicate which by writing a 'yes' or a 'no' in the appropriate blank.

Remember! If you say 'yes' under 'Necessary' then you are claiming that

if (*) then [......]

is a theorem.

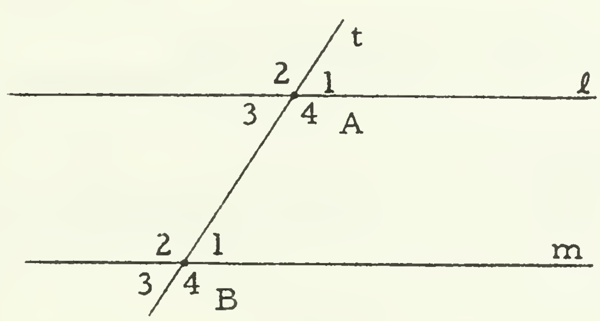
If you say 'yes' under 'Sufficient' then you are claiming that

if [......] then (*)

is a theorem.

Also, be prepared to state the appropriate conditional for each sentence which gets just one 'yes'-answer, and the appropriate biconditional for each sentence which gets two 'yes'-answers.

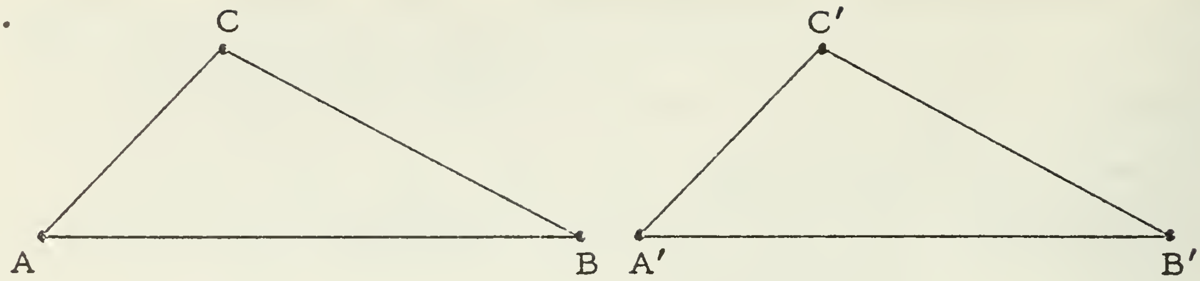
1.



(*) ℓ is parallel to m

	Necessary for (*)	Sufficient for (*)
(a) $\angle A_1 \cong \angle B_1$	______	______
(b) $m(\angle A_1) = 40 = m(\angle B_1)$	______	______
(c) $m(\angle A_3) + m(\angle B_2) = 180$	______	______
(d) $t \perp \ell$ and $t \perp m$	______	______
(e) $\ell \cap m = \emptyset$	______	______
(f) $\angle A_2 \cong \angle A_4$ and $\angle B_2 \cong \angle B_4$	______	______
(g) ℓ is not perpendicular to m	______	______

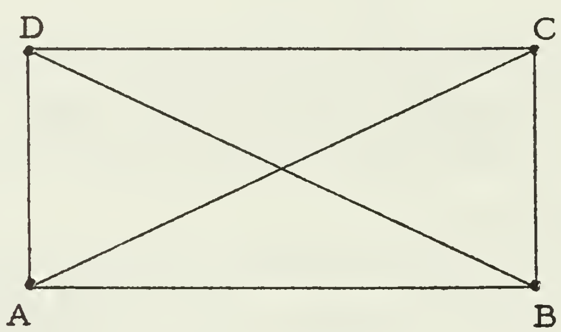
2.



(*) $\angle A \cong \angle A'$ and $\angle B \cong \angle B'$

	Necessary for (*)	Sufficient for (*)
(a) $ABC \leftrightarrow A'B'C'$ is a congruence	______	______
(b) $ABC \leftrightarrow A'C'B'$ is a congruence	______	______
(c) $ABC \leftrightarrow C'A'B'$ is a congruence	______	______
(d) the two triangles are congruent	______	______
(e) $\angle C \cong \angle C'$	______	______
(f) $AB = A'B'$, $AC = A'C'$, and $BC = B'C'$	______	______

3.

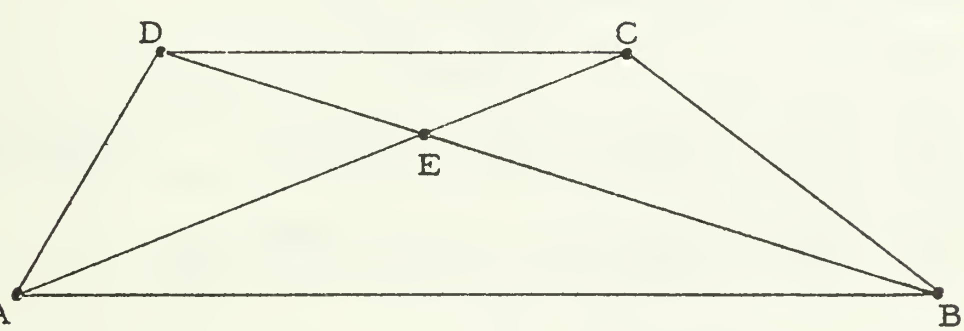


(*) ABCD is a rectangle

	Necessary for (*)	Sufficient for (*)
(a) ABCD is a parallelogram	______	______
(b) ABCD is a square	______	______
(c) $AC = BD$	______	______
(d) $AC = BD$ and $\overline{BD}$ bisects $\overline{AC}$	______	______

		Necessary for (*)	Sufficient for (*)
(e)	the midpoint of $\overline{BD}$ is equidistant from A, B, and C	______	______
(f)	$\angle DAB$ and $\angle ABC$ are right angles	______	______
(g)	$\angle DAB$ and $\angle ABC$ and $\angle BCD$ are right angles	______	______
(h)	$ADB \leftrightarrow BCA$ is a congruence	______	______
(i)	$ADB \leftrightarrow BCA$ is a congruence and $\angle DAB$ is a right angle	______	______
(j)	$\angle DAB \cong \angle ABC \cong \angle BCD \cong \angle CDA$	______	______
(k)	$\angle DAC \cong \angle CAB \cong \angle ABD \cong \angle DBC \cong \angle BCA \cong \angle ACD \cong \angle CDB \cong \angle BDA$	______	______
(ℓ)	$\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$ and $\angle DAB \cong \angle ABC \cong \angle BCD$	______	______
(m)	ABCD is a parallelogram and $AC = BD$	______	______
(n)	$\overline{AC} \cong \overline{BD}$ and $\overline{AC}$ and $\overline{BD}$ bisect each other	______	______

4.

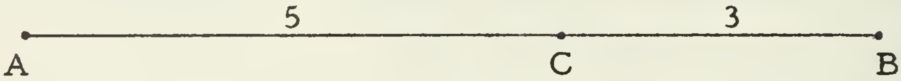


(*) ABCD is a trapezoid

		Necessary for (*)	Sufficient for (*)
(a)	$\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$	______	______
(b)	$\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$ and $\overleftrightarrow{AD} \nparallel \overleftrightarrow{BC}$	______	______
(c)	$\overleftrightarrow{AB} \nparallel \overleftrightarrow{DC}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$	______	______
(d)	$\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$ and $AE \neq EC$	______	______

EXPLORATION EXERCISES

A. Here is a picture of $\overline{AB}$ and a point C which divides $\overline{AB}$ into two segments with a common end point:



We say that C divides the segment from A to B in a certain ratio, this ratio being the quotient of m($\overline{AC}$) by m($\overline{CB}$). In the case shown, C divides the segment from A to B in the ratio 5/3, or 5 : 3, [or: 5 to 3].

Of course, C also divides the segment from B to A in the ratio 3/5, or 3 : 5.

1.

Suppose that AB = BC = CD = DE = EF, as shown. Complete each of the following sentences.

(a) B divides the segment from A to D in the ratio ______.

(b) B divides the segment from D to A in the ratio ______.

(c) D divides the segment from C to E in the ratio ______.

(d) The ratio of $\overline{AC}$ to $\overline{CF}$ is ______.

(e) The ratio of $\overline{AC}$ to $\overline{EF}$ is ______.

(f) If $\frac{AB}{MN} = \frac{CD}{JK}$ then $\overline{MN}$ and $\overline{JK}$ are _______ segments.

(g) If $\frac{AB}{MN} = \frac{CE}{JK}$ then the ratio of $\overline{MN}$ to $\overline{JK}$ is ______ .

2. Suppose that $P \in \overline{MN}$.

(a) If P divides the segment from M to N in the ratio 2 : 3, and MN = 20, then MP = ______ and PN = ______.

(b) If P is the midpoint of $\overline{MN}$ then the ratio of $\overline{MP}$ to $\overline{PN}$ is ______ and the ratio of $\overline{MP}$ to $\overline{MN}$ is ______.

(c) If the ratio of $\overline{MP}$ to $\overline{MN}$ is 3 : 5 then P divides the segment from M to N in the ratio ______.

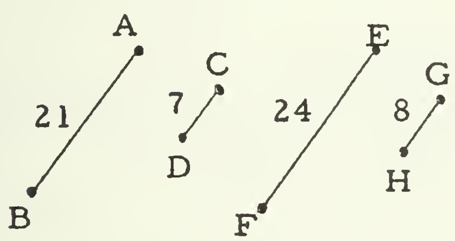
3. Suppose that $P \in \overline{MN}$ and that $Q \in \overline{RS}$.

   (a) P divides the segment from M to N in the ratio 4 : 5. If Q divides the segment from R to S in the same ratio and $\overset{\bullet\!-\!\bullet}{MP} \cong \overset{\bullet\!-\!\bullet}{RQ}$, then ____ $\cong$ ____.

   (b) If the ratio of $\overset{\bullet\!-\!\bullet}{MP}$ to $\overset{\bullet\!-\!\bullet}{PN}$ is equal to the ratio of $\overset{\bullet\!-\!\bullet}{RQ}$ to $\overset{\bullet\!-\!\bullet}{QS}$ and $MP = 2 \cdot RQ$, the ratio of $\overset{\bullet\!-\!\bullet}{PN}$ to $\overset{\bullet\!-\!\bullet}{QS}$ is ______.

B. In doing the preceding exercises, you probably guessed what is meant by <u>the ratio of a first segment to a second segment.</u> The ratio of such segments [nondegenerate ones, of course] is the quotient of the measure of the first by the measure of the second.

If the ratio of a first segment to a second segment is the same as the ratio of a third segment to a fourth then the segments, in that order, are said to be <u>in proportion.</u> For example,

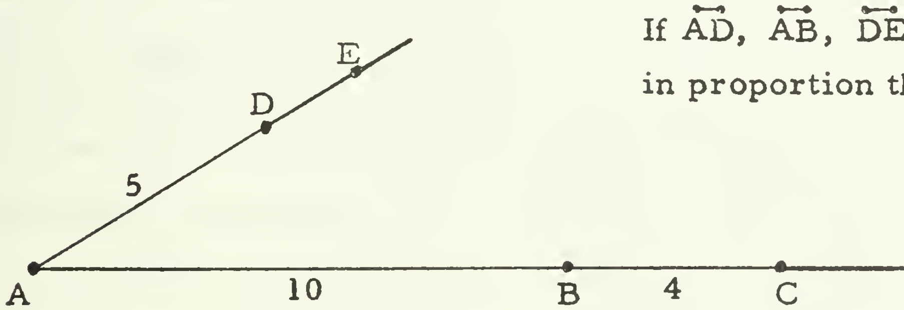


$\overset{\bullet\!-\!\bullet}{AB}$, $\overset{\bullet\!-\!\bullet}{CD}$, $\overset{\bullet\!-\!\bullet}{EF}$, and $\overset{\bullet\!-\!\bullet}{GH}$ are in proportion because

$$\frac{AB}{CD} = \frac{EF}{GH}.$$

Also, $\overset{\bullet\!-\!\bullet}{AB}$, $\overset{\bullet\!-\!\bullet}{EF}$, $\overset{\bullet\!-\!\bullet}{CD}$, and $\overset{\bullet\!-\!\bullet}{GH}$ are in proportion. Why?
But, $\overset{\bullet\!-\!\bullet}{AB}$, $\overset{\bullet\!-\!\bullet}{EF}$, $\overset{\bullet\!-\!\bullet}{GH}$, and $\overset{\bullet\!-\!\bullet}{CD}$ are not in proportion. Why not?

1. For the figure above, $\overset{\bullet\!-\!\bullet}{AB}$, ___, ___, and ___ are also in proportion, but $\overset{\bullet\!-\!\bullet}{AB}$, ___, ___, and ___ are not in proportion.

2. 

   If $\overset{\bullet\!-\!\bullet}{AD}$, $\overset{\bullet\!-\!\bullet}{AB}$, $\overset{\bullet\!-\!\bullet}{DE}$, and $\overset{\bullet\!-\!\bullet}{BC}$ are in proportion then DE = ____.

3. For the figure in Exercise 2, if $\overset{\bullet\!-\!\bullet}{AD}$, $\overset{\bullet\!-\!\bullet}{DE}$, $\overset{\bullet\!-\!\bullet}{AB}$, and $\overset{\bullet\!-\!\bullet}{AC}$ are in proportion, does it follow that $\overset{\bullet\!-\!\bullet}{AD}$, $\overset{\bullet\!-\!\bullet}{AE}$, $\overset{\bullet\!-\!\bullet}{AB}$, and $\overset{\bullet\!-\!\bullet}{AC}$ are in proportion?

4. Suppose that $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{EF}$ are in proportion and that $\overline{CD}$, $\overline{EF}$, $\overline{FG}$, and $\overline{GH}$ are in proportion. Prove that $\overline{AB}$, $\overline{BC}$, $\overline{FG}$, and $\overline{GH}$ are in proportion.

<u>C.</u> If $\overline{AB}$, $\overline{CD}$, $\overline{EF}$, and $\overline{GH}$ are in proportion, one sometimes says that $\overline{AB}$ and $\overline{EF}$ <u>are proportional to</u> $\overline{CD}$ and $\overline{GH}$. Since this amounts to saying that

$$\frac{AB}{CD} = \frac{EF}{GH},$$

it is easy to see how to generalize this idea. For example, to say that

$\overline{JK}$, $\overline{LM}$, and $\overline{NP}$ are proportional to $\overline{QR}$, $\overline{ST}$, and $\overline{UV}$

is to say that

$$\frac{JK}{QR} = \frac{LM}{ST} = \frac{NP}{UV}.$$

1. If $\overline{A_1A_2}$, $\overline{B_1B_2}$, and $\overline{C_1C_2}$ are proportional to $\overline{D_1D_2}$, $\overline{E_1E_2}$, and $\overline{F_1F_2}$, and the ratio of $\overline{A_1A_2}$ to $\overline{D_1D_2}$ is 4 : 5, the ratio of $\overline{B_1B_2}$ to $\overline{E_1E_2}$ is _____, and the ratio of $\overline{F_1F_2}$ to $\overline{C_1C_2}$ is _____.

2. Suppose that $\overline{M_1M_2}$ and $\overline{N_1N_2}$ are proportional to $\overline{N_1N_2}$ and $\overline{P_1P_2}$. If $M_1M_2 = 4$ and $P_1P_2 = 9$ then $N_1N_2 =$ _____.

*

Since a measure is a number, the ideas of ratio and proportion for segments are just applications of notions of ratio and proportion for numbers. In general, for nonzero numbers x, y, u, and v, the ratio of x to y is $x \div y$ and x, y, u, and v are in proportion if and only if $x \div y = u \div v$. In this case, we also say that x and u are proportional to y and v. Similarly, for any sequences of nonzero numbers x_1, x_2, x_3, ... and y_1, y_2, y_3, ... ,

$x_1, x_2, x_3, \ldots$ is proportional to $y_1, y_2, y_3, \ldots$

if and only if

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3} = \cdots .$$

*

3. Solve.

   (a) 3, 5 is proportional to 6, x (b) 8, x is proportional to 12, 6

   (c) 2, 7, x is proportional to 10, 35, 50

   (d) 1, 2, 3 is proportional to 10, x, 30

   (e) 5, x is proportional to x, 20 (f) 3, $x + 5$ is proportional to 8, $x - 2$

4. If 3, 5, 9 is proportional to a sequence one of whose terms is 10, what is the sequence?

5. If a, b, c is proportional to a sequence whose first term is ak, what is the sequence?

6. If a, b, c is proportional to a sequence whose first term is d, what is the sequence?

7. Suppose you multiply each term in a sequence of nonzero numbers by a nonzero number. Is the first sequence proportional to the second?

8. $x_1, x_2, x_3, \ldots$ is proportional to $y_1, y_2, y_3, \ldots$ if and only if there is a nonzero number k such that $y_1 =$ ____, $y_2 =$ ____, $y_3 =$ ____, ...

9. Show that if a, b is proportional to c, d then b, a is proportional to d, c.

10. Justify each of the following.

    (a) If a, b is proportional to c, d then $ad = bc$.

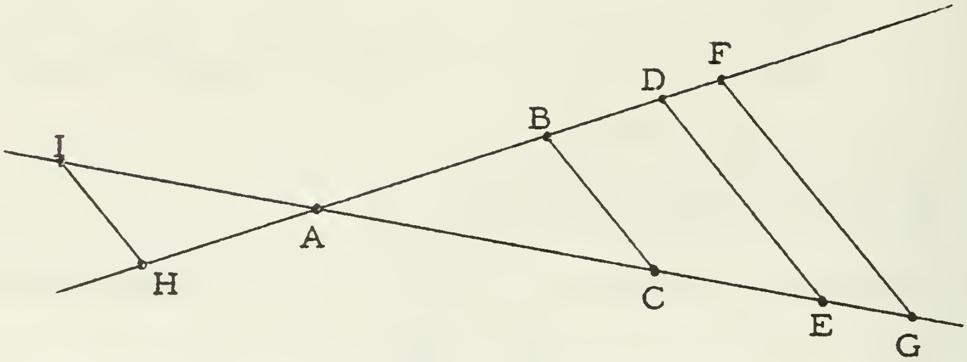
    (b) If $ad = bc$ then a, b is proportional to c, d.

    (c) If a, b is proportional to c, d then a, c is proportional to b, d.

    (d) If a, b is proportional to c, d then a, b is proportional to $a + c$, $b + d$.

    (e) If a, b is proportional to c, d then a, $a + b$ is proportional to c, $c + d$.

11. <u>Given:</u> $\overline{HA}$, $\overline{AB}$, $\overline{BD}$, $\overline{DF}$ are proportional to $\overline{IA}$, $\overline{AC}$, $\overline{CE}$, $\overline{EG}$



Fill in the blanks and use the theorems in Exercise 10 to justify your choice.

1. $\frac{AB}{AC} = \frac{AD}{\_\_\_}$ 2. $\frac{AB}{AD} = \frac{\_\_\_}{AE}$ 3. $\frac{AH}{HB} = \frac{AI}{\_\_\_}$

4. $\frac{\_\_\_}{BF} = \frac{AE}{CG}$ 5. $\frac{FD}{\_\_\_} = \frac{EG}{CI}$ 6. $\frac{BF}{\_\_\_} = \frac{\_\_\_}{EG}$

7. $AB \cdot CE = AC \cdot$ _____ 8. $AF \cdot AC =$ ____ $\cdot AB$

9. $CG \cdot AH =$ ____ $\cdot$ ____ 10. $AB \cdot AI =$ ____ $\cdot$ ____

12. In Exercises 10(a) and 10(b) you proved that

'if a, b is proportional to c, d then ad = bc'

and

'if ad = bc then a, b is proportional to c, d'

are theorems.

Make four statements like (I), (II), (III), and (IV) on page 6-181.

13. From your work in Exercise 10 complete the following:

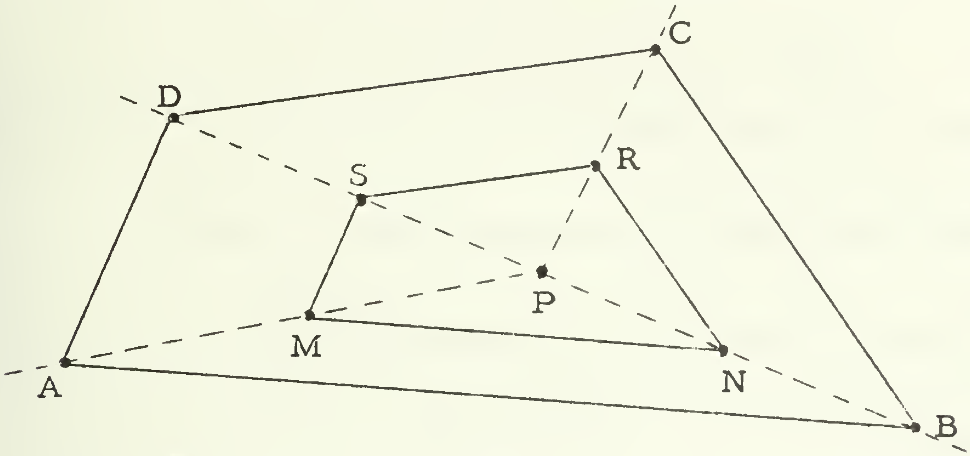
(a) 'a, b is proportional to c, d' is a __________ condition for 'a, c is proportional to b, d'.

(b) 'a, b is proportional to c, d _______ a, b is proportional to a + c, b + d' is a theorem.

(c) 'a, a + b is proportional to c, c + d' is a __________ condition for 'a, b is proportional to c, d'.

14. State the converses of the conditionals in Exercises 10(c), (d), and (e). Are the converses theorems?

6.07 <u>Similar polygons.</u>--Here is a quadrilateral ABCD, and a point P. The points M, N, R, and S are the midpoints of $\overleftrightarrow{PA}$, $\overleftrightarrow{PB}$, $\overleftrightarrow{PC}$, and $\overleftrightarrow{PD}$.



Notice that $\overleftrightarrow{MN}$ is parallel to $\overleftrightarrow{AB}$ and that $MN = \frac{1}{2} \cdot AB$ [Why?]. Similarly, $\overleftrightarrow{NR}$ is parallel to $\overleftrightarrow{BC}$ and $NR = \frac{1}{2} \cdot BC$. Etc.

So, $\angle SMN \cong \angle DAB$, $\angle MNR \cong \angle ABC$, $\angle NRS \cong \angle BCD$, and $\angle RSM \cong \angle CDA$. [Why?]

And, $\overleftrightarrow{MN}$, $\overleftrightarrow{NR}$, $\overleftrightarrow{RS}$, and $\overleftrightarrow{SM}$ are proportional to $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{DA}$. [Why?]

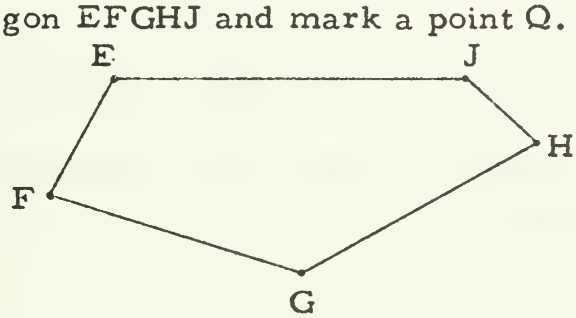
Thus, with respect to the matching

$$MNRS \longleftrightarrow ABCD,$$

the angles of quadrilateral MNRS are congruent to the corresponding angles of quadrilateral ABCD, and the sides of quadrilateral MNRS are proportional to the corresponding sides of quadrilateral ABCD.

EXERCISES

<u>A.</u> Draw the pentagon EFGHJ and mark a point Q. Use the technique



shown above to draw a pentagon E′F′G′H′J′ such that with respect to the matching

$$EFGHJ \longleftrightarrow E'F'G'H'J',$$

the corresponding angles are congruent and the corresponding sides are proportional.

B. Draw a rectangle ABCD with AB = 2 and BC = 4. Draw a segment $\overleftrightarrow{EF}$ such that $\overleftrightarrow{EF}$ is parallel to none of the sides of rectangle ABC and such that EF = 6. Now, draw a quadrilateral EFGH such that with respect to the matching EFGH $\longleftrightarrow$ ABCD, the corresponding angles are congruent and the corresponding sides are proportiona

C. Repeat Part B for the matching EGHE $\longleftrightarrow$ ABCD.

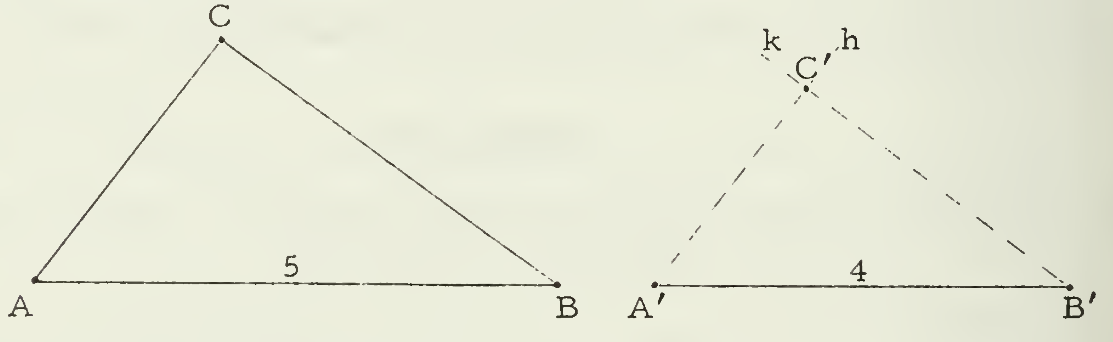
[Supplementary Exercises are on page 6-430.]

SIMILARITY

A matching of the vertices of a first polygon with those of a secon for which corresponding angles are congruent and corresponding sides are proportional is called a similarity. We say that a first polygon is similar to a second if and only if there is a matching of their vertices which is a similarity.

Roughly speaking, two polygons are similar if and only if they hav the same shape. Examples of similarity in everyday life occur in making enlargements of photographs and in working with scale drawin and blueprints.

Consider the triangle $\triangle ABC$ and the segment $\overleftrightarrow{A'B'}$. Suppose that



wish to draw a triangle $\triangle A'B'C'$ such that $A'B'C' \longleftrightarrow ABC$ is a simila Now, by the definition of similarity, this means that in the required triangle,

$$\angle A' \cong \angle A, \qquad \angle B' \cong \angle B, \qquad \angle C' \cong \angle C,$$

and

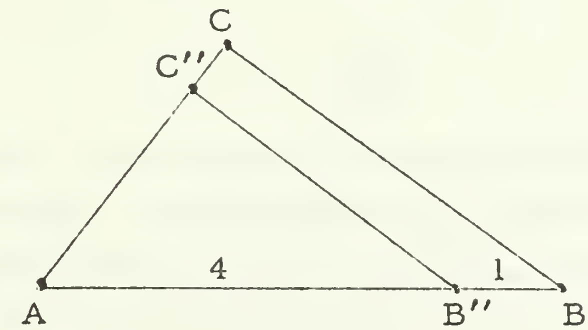
$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA}.$$

[In this case, the ratio of corresponding sides is 4:5.]

Let t be the "upper" one of the two half-planes determined by the line through A' and B'. By our axioms, we know that there is a unique half-line h contained in t such that $h \cup \overrightarrow{A'B'} \cong \angle A$. Similarly, there is a unique half-line k contained in t such that $k \cup \overrightarrow{B'A'} \cong \angle B$.

Since $\angle A$ and $\angle B$ are not supplementary, h and k intersect in a point C'. Now, $\triangle A'B'C'$ satisfies the similarity condition about congruent corresponding angles. [How do you know that $\angle C' \cong \angle C$?] But, does it satisfy the other similarity condition, the one about proportional corresponding sides? It does. Let's prove it.

Since $A'B' < AB$, there is a point $B'' \in \overline{AB}$ such that $AB'' = A'B'$. Also, there is a point $C'' \in \overrightarrow{AC}$ such that $AC'' = A'C'$.



Now, $AB''C'' \longleftrightarrow A'B'C'$ is a congruence. [Why?] So, since $\angle AB''C'' \cong \angle B'$ and $\angle B' \cong \angle B$, it follows that $\overleftrightarrow{B''C''} \parallel \overleftrightarrow{BC}$.

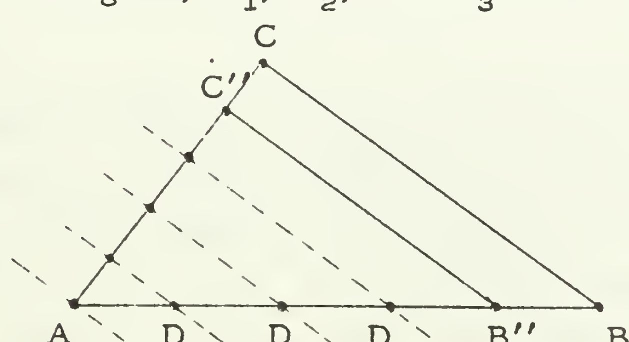
We know that

$$\frac{A'B'}{AB} = \frac{4}{5},$$

and that $AB'' = A'B'$.

So, $AB'' = 4 \cdot B''B$.

Now, we can find three points, D_1, D_2, and D_3, in $\overline{AB''}$ such that $AD_1 = D_1D_2 = D_2D_3 = D_3B'' = B''B$. Let's mark these points, and consider the lines through A, D_1, D_2, and D_3 which are parallel to $\overleftrightarrow{B''C''}$.



These four lines together with $\overleftrightarrow{B''C''}$ and $\overleftrightarrow{BC}$ are six parallel lines

which cut off five congruent segments on the transversal $\overleftrightarrow{AB}$. So, by Theorem 6-27, these parallel lines cut off five congruent segments on the transversal $\overleftrightarrow{AC}$. So,

$$\frac{C''A}{CA} = \frac{4}{5}.$$

But, since $C''A = C'A'$,

$$\frac{C'A'}{CA} = \frac{4}{5}.$$

A similar argument, based on considering the point $A'' \in \overline{AB}$ such that $BA'' = B'A'$, and the point $C''' \in \overrightarrow{BC}$ such that $BC''' = B'C'$ shows that

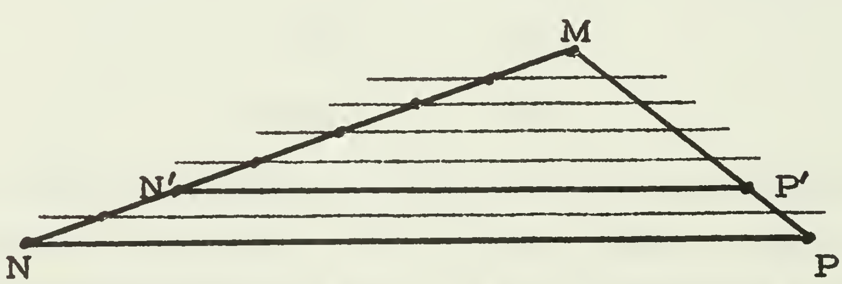
$$\frac{B'C'}{BC} = \frac{4}{5}.$$

[Go through the details of this argument.]

This completes the proof that $A'B'C' \longleftrightarrow ABC$ is a similarity.

One result of this argument is that if, in $\triangle A'B'C'$ and $\triangle ABC$, $\angle A' \cong \angle A$, $\angle B' \cong \angle B$, $\angle C' \cong \angle C$, and $A'B'/AB = 4/5$, then $A'B'C' \longleftrightarrow ABC$ is a similarity.

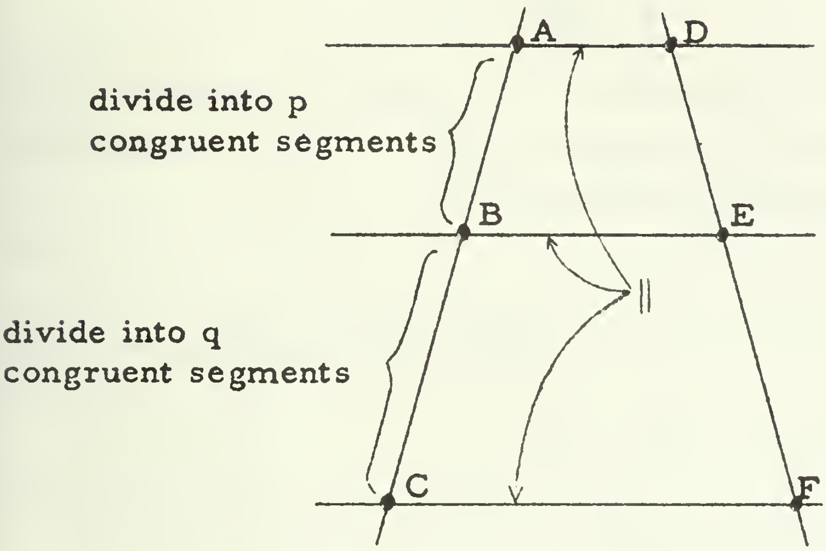
Now, suppose you have two triangles, $\triangle MNP$ and $\triangle MN'P'$, for which $\angle M \cong \angle M$, $\angle N \cong \angle MN'P'$, and $MN'/MN = 5/7$. The same kind of



reasoning could be used to show that $\angle P \cong \angle N'P'M$, that $MP'/MP = 5/7$, and that $N'P'/NP = 5/7$. So, $MNP \longleftrightarrow MN'P'$ is a similarity.

In general, if two triangles agree in their angles, and the ratio of a pair of corresponding sides is m/n, where m and n are nonzero <u>whole</u> numbers, then the ratio of each pair of corresponding sides is m/n. So, by definition, the triangles are similar.

The same argument shows that if three parallel lines cut off, on



Given: $\frac{AB}{BC} = \frac{p}{q}$

Conclusion: $\frac{DE}{EF} = \frac{p}{q}$

one transversal, two segments whose ratio is p/q, where p and q are nonzero whole numbers, then the ratio of the segments which they cut off on any other transversal is also p/q. [Go through the argument once more.]

Let us examine the restriction in this theorem, the restriction that p and q be whole numbers. Does this mean that the measures of the segments must be whole numbers? Suppose that, in the figure above, AB = 3/4 and BC = 3/2. Then, AB/BC = 1/2. So, we can take p to be 1 and q to be 2. [What are other values of 'p' and 'q' which fit?] The theorem tells us that DE/EF = 1/2.

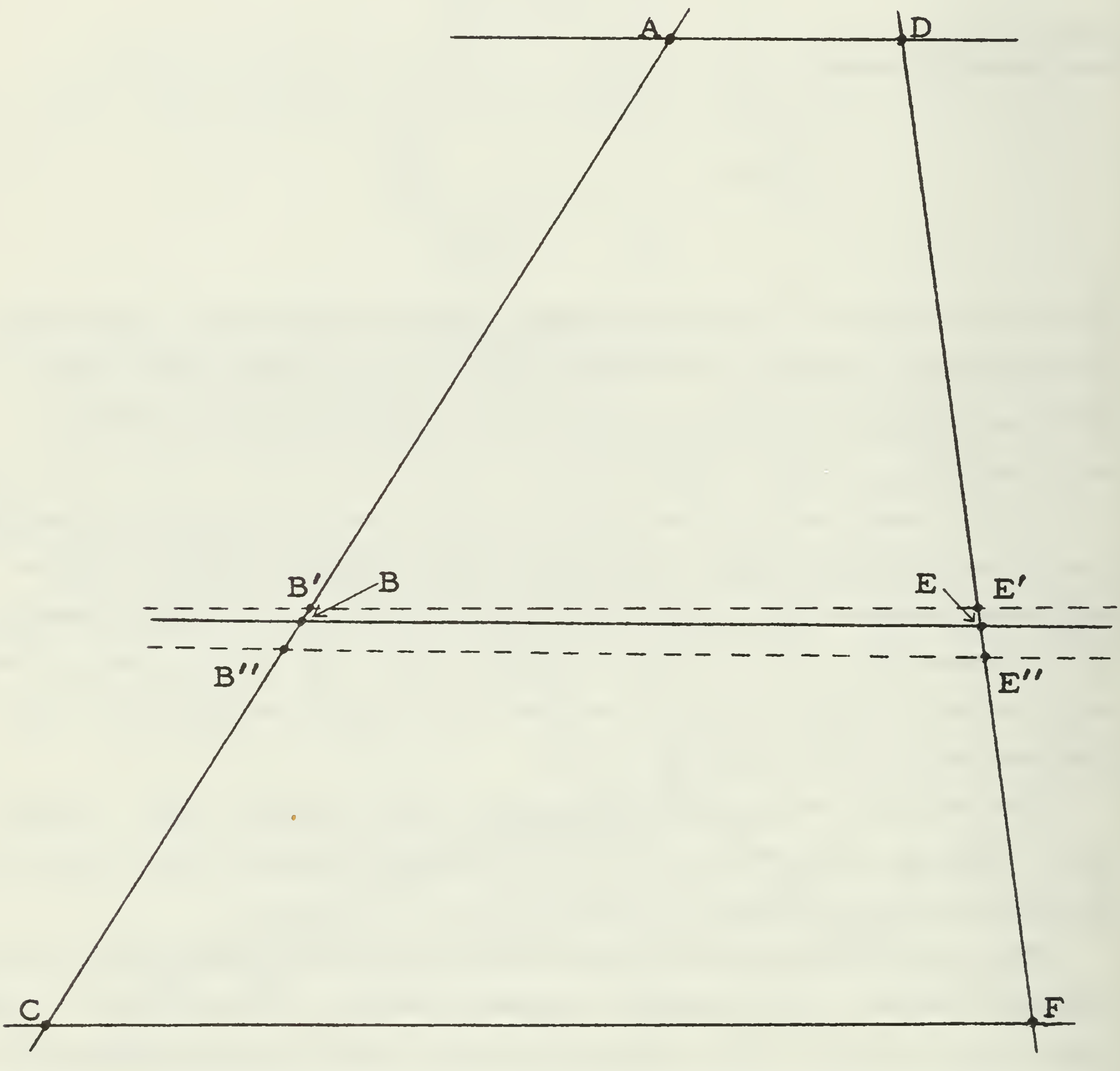
The theorem would also apply if AB = $2\sqrt{2}$ and BC = $3\sqrt{2}$. In this case, although neither AB nor BC is a rational number, their ratio <u>is</u> a quotient of whole numbers. What is it?

But, now, consider the case in which the ratio of AB to BC is not a rational number. For example, suppose that AB = $\sqrt{2}$ and BC = 1. The ratio of AB to BC is $\sqrt{2}/1$, and in an earlier unit you proved that $\sqrt{2}$ is not a rational number--that is, there are <u>not</u> two whole numbers, p and q, such that $q\sqrt{2} = p$. So, the argument we have been using doesn't apply because we cannot divide $\overleftrightarrow{AC}$ into a whole number of congruent segments in such a way that B is one of the division points. [Explain.] Nevertheless, it can be shown by a more complicated argument that AB/BC = DE/EF. Here is how.

We are given that $AB = \sqrt{2}$ and $BC = 1$. Since, as you will recall, $1.4 < \sqrt{2} < 1.5$, it follows that

$$(*) \quad 1.4 < \frac{AB}{BC} < 1.5.$$

Choose a point $B' \in \overrightarrow{AC}$ such that $AB' = 1.4$. Then, by Theorem 1-5, $B' \in \overline{AB}$. [Explain.] Let E' be the point of $\overline{DE}$ in which the parallel to $\overleftrightarrow{AD}$ through B' intersects the transversal $\overleftrightarrow{DF}$.



Since $AB'/BC = 1.4/1 = 14/10$, we can divide $\overline{AB'}$ and $\overline{BC}$, respectively, into 14 segments and 10 segments all of which are congruent. So, the lines parallel to $\overleftrightarrow{AD}$, through the division points divide $\overline{DE'}$ and $\overline{EF}$, respectively, into 14 segments and 10 segments all of which are congruent. So, $DE'/EF = 14/10$. Since $DE' < DE$, it follows that $DE'/EF < DE/EF$. Hence, $1.4 < DE/EF$.

Now, also, there is a point $B'' \in \overrightarrow{AC}$ such that $AB'' = 1.5$. This point, B'', belongs to $\overline{BC}$. If the parallel to $\overleftrightarrow{AD}$ through B'' intersects $\overleftrightarrow{DF}$ in E'' then, by a similar argument, we can show that $DE''/EF = 15/10$. Since $DE < DE''$, it follows that $DE/EF < 1.5$.

So, we have shown that

$$1.4 < \frac{DE}{EF} < 1.5.$$

This implies that DE/EF differs from $\sqrt{2}$ by less than 0.1. [Explain.]

Now, using the fact that $1.41 < \sqrt{2} < 1.42$, we can show by an entirely similar argument that

$$1.41 < \frac{DE}{EF} < 1.42.$$

And, this implies that DE/EF differs from $\sqrt{2}$ by less than 0.01.

By using a sufficiently close rational approximation to $\sqrt{2}$, we can show that DE/EF differs from $\sqrt{2}$ by as little as we please. But, this can be so if and only if DE/EF differs from $\sqrt{2}$ by 0. So,

$$\frac{DE}{EF} = \frac{\sqrt{2}}{1} = \frac{AB}{BC}.$$

Since each number of arithmetic can be approximated as closely as one wishes by rational numbers, the preceding argument can be generalized to show that no matter what the ratio of $\overset{\bullet\!-\!\bullet}{AB}$ to $\overset{\bullet\!-\!\bullet}{BC}$ is,

$$\frac{AB}{BC} = \frac{DE}{EF}.$$

Let us summarize this in the following theorem:

<u>Theorem 7-1.</u>

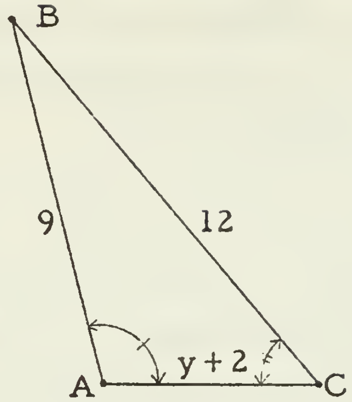
Three parallel lines cut off proportional segments on each two transversals.

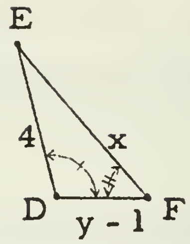
We also have the following theorem on triangle-similarity:

<u>Theorem 7-2.</u> [a. a. similarity theorem]

If, for some matching of the vertices of one triangle with those of a second, each of two angles of the first triangle is congruent to the corresponding angle of the second, then the matching is a similarity.

Example 1. Find: EF, AC, DF





Since $\angle A \cong \angle D$ and $\angle C \cong \angle F$, it follows from the a.a. similarity theorem that $ABC \longleftrightarrow DEF$ is a similarity. Hence, by the definition of similarity,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$$

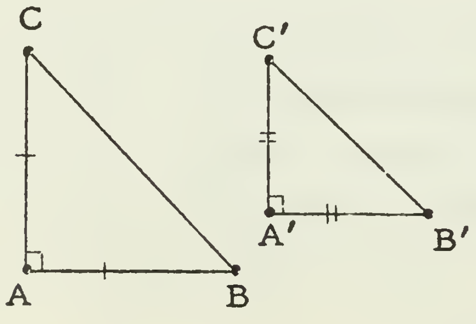
So, $$\frac{9}{4} = \frac{12}{x} \quad \text{and} \quad \frac{9}{4} = \frac{y+2}{y-1}.$$

We solve each of these proportions.

$\frac{9}{4} = \frac{12}{x}$	$\frac{9}{4} = \frac{y+2}{y-1}$
$9x = 48$	$9(y-1) = 4(y+2)$
$x = \frac{16}{3}$	$9y - 9 = 4y + 8$
	$5y = 17$
	$y = \frac{17}{5}$

Therefore, $EF = \frac{16}{3}$, $AC = \frac{27}{5}$, and $DF = \frac{12}{5}$.

Example 2. Given that ΔABC and $\Delta A'B'C'$ are isosceles triangles with right angles, $\angle A$ and $\angle A'$. Show that $\Delta ABC \sim \Delta A'B'C'$. ['$\sim$' is read as 'is similar to'.]



Hypothesis: $\angle A$ and $\angle A'$ are angles of 90°
$AB = AC$,
$A'B' = A'C'$

Conclusion: $\Delta ABC \sim \Delta A'B'C'$

Plan. Let's try to show that $\triangle ABC \sim \triangle A'B'C'$ by Theorem 7-2. By the hypothesis, $\angle A$ and $\angle A'$ are right angles; so, we know that $\angle A \cong \angle A'$.

What do we know about $\angle B$? By the hypothesis, $AB = AC$. So, $\angle B \cong \angle C$. But, $\angle A$ is a right angle; so, we know that $m(\angle B) + m(\angle C) = 90$. [Why?] Hence, $m(\angle B) = 45$. Similarly, we see that $m(\angle B') = 45$. So, we have $\angle B \cong \angle B'$. By Theorem 7-2, $ABC \longleftrightarrow A'B'C'$ is a similarity.

Solution.

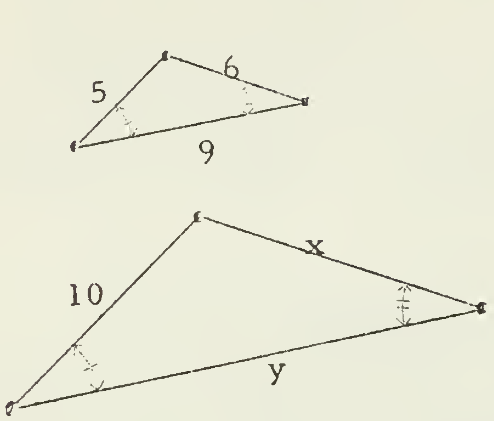
(1)	A, B, C and A′, B′, C′ are vertices of triangles	[figure]
(2)	$AB = AC$	[Hypothesis]
(3)	Two sides of a triangle are congruent if and only if the angles opposite them are congruent.	[theorem]
(4)	$\angle C \cong \angle B$	[(1), (2), and the only-if-part of (3)]
(5)	The sum of the measures of the angles of a triangle is 180.	[theorem]
(6)	$m(\angle A) + m(\angle B) + m(\angle C) = 180$	[(1) and (5)]
(7)	$m(\angle A) = 90$	[Hypothesis]
(8)	$m(\angle B) + m(\angle C) = 90$	[(6) and (7)]
(9)	$m(\angle B) = 45$	[(4) and (8); def. of congruent angles]
(10)	$m(\angle B') = 45$	[Steps like (1) - (9)]
(11)	$\angle B \cong \angle B'$	[(9) and (10); def. of congruent angles]
(12)	$\angle A \cong \angle A'$	[Hypothesis; def. of congruent angles]
(13)	a.a. similarity theorem	[theorem]
(14)	$ABC \longleftrightarrow A'B'C'$ is a similarity	[(1), (11), (12), and (13)]
(15)	$\triangle ABC \sim \triangle A'B'C'$	[(14); def. of similar triangles]

Note. How would you continue the proof to show that $AB \cdot A'C' = A'B' \cdot AC$?

EXERCISES

A. Give the indicated measures.

1.



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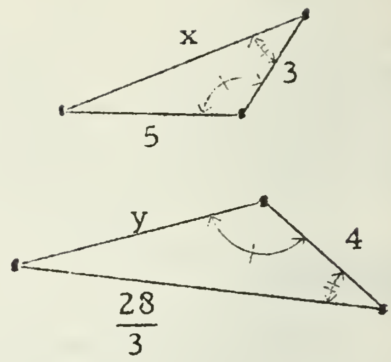

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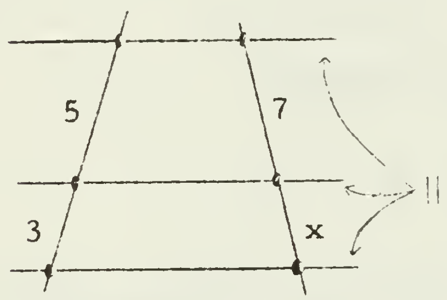

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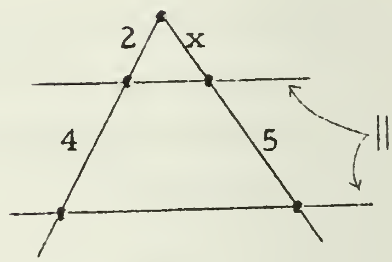

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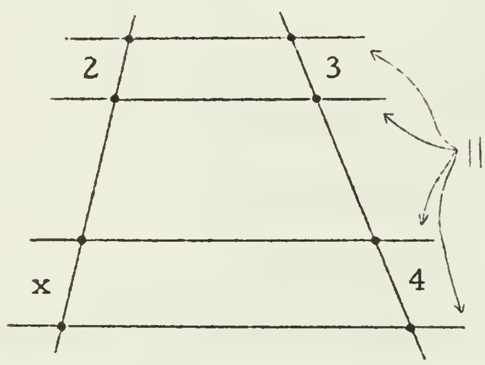

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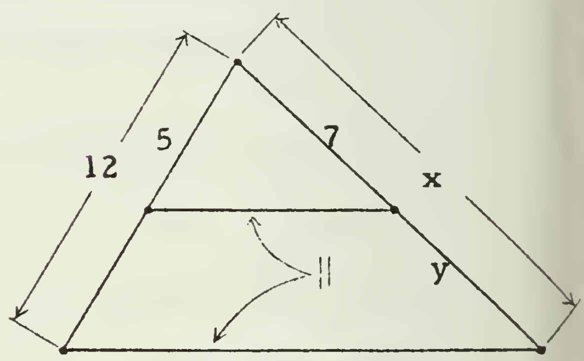

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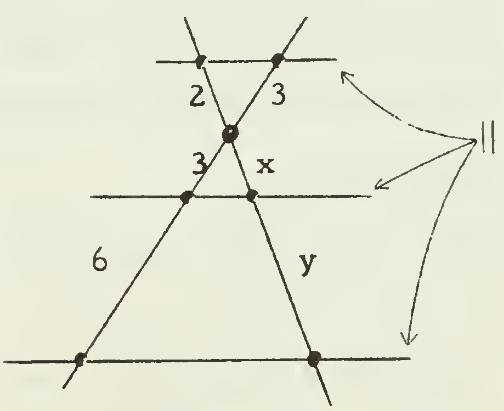

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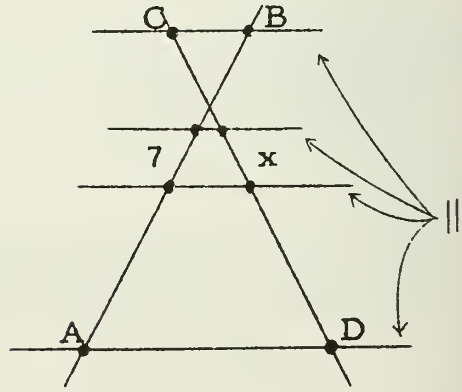


AB = 35

CD = 30

B. Suppose that $\Delta ABC \sim \Delta DEF$. [In particular, suppose that $ABC \leftrightarrow DEF$ is a similarity.]

1. If $AB = 2$, $BC = 7$, $CA = 6$, and $EF = 14$, what are FD and DE?

2. If $AB = 4$, $BC = 10$, and $DE = 7$, what is EF?

3. If $BC = 4$, $EF = 6$, $AB = DE - \frac{3}{2}$, $FD = AC + \frac{5}{2}$, what are AB, AC, DE, and FD?

C. Suppose that $MNP \leftrightarrow SQR$ is a similarity, that $\overleftrightarrow{MK}$ is the altitude of ΔMNP from M, and that $\overleftrightarrow{ST}$ is the altitude of ΔSQR from S. If $MN = 13$, $SQ = 26$, $MK = 12$, $QT = 10 = KP$, what are ST, NK, and TR?

D. Given equilateral triangles, ΔABC and ΔDEF. Show that $\Delta ABC \sim \Delta DEF$.

E. Given that ΔABC and $\Delta A'B'C'$ are right-angled at A and A′, respectively, and that $\angle B \cong \angle B'$. Show that the triangles are similar.

* * *

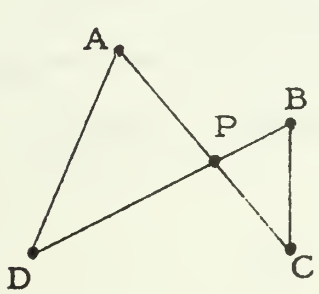
Your work in Part E leads to the following theorem:

Theorem 7-3.

If two right triangles agree in a pair of acute angles, the triangles are similar.

* * *

F. 1.



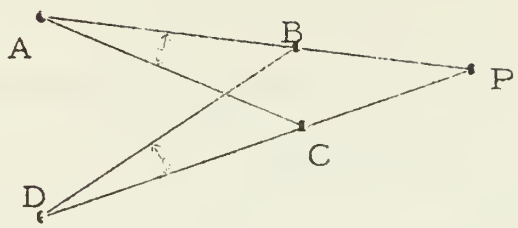
Hypothesis: $\angle D \cong \angle C$, $P \in \overline{BD}$, $P \in \overline{AC}$

Conclusion: $APD \leftrightarrow BPC$ is a similarity,

$\frac{AP}{BP} = \frac{PD}{PC}$,

$AP \cdot PC = BP \cdot PD$

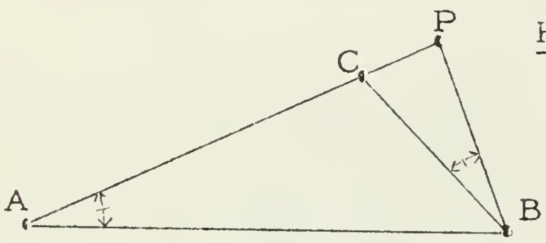
2.



Hypothesis: $\angle A \cong \angle D$

Conclusion: $PA \cdot PB = PD \cdot PC$

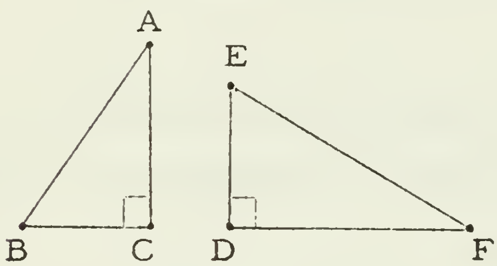
3.



Hypothesis: $\angle A \cong \angle PBC$

Conclusion: $\Delta PBC \sim \Delta PAB$,
$(PB)^2 = PA \cdot PC$

G. 1.

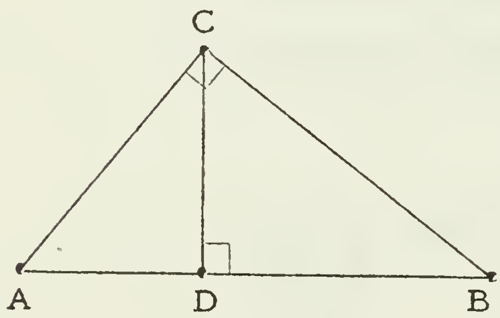


Hypothesis: $\angle C$ and $\angle D$ are right angles,
$\angle B$ and $\angle F$ are complementary

Conclusion: $\Delta ABC \sim \Delta FED$

2. In Exercise 1, if $AB = 5$, $BC = 3$, $CA = 4$, and $EF = 10$, what are ED and DF?

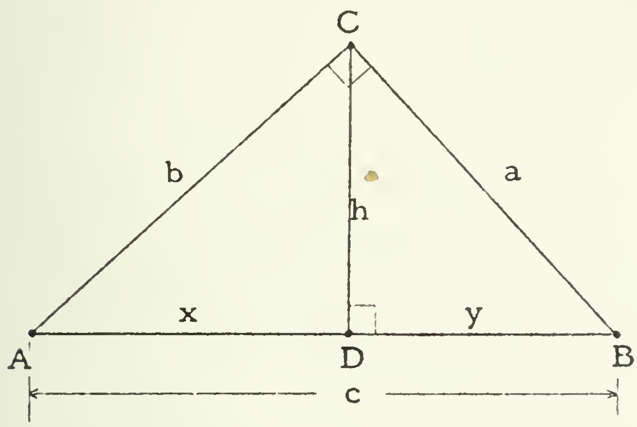
H.



1. Show that $\Delta ADC \sim \Delta CDB$.
2. If $AD = 3$ and $DB = 12$, what is CD?
3. Show that $\Delta ACD \sim \Delta ABC$.
4. If $AD = 5$ and $AB = 20$, what is AC?
5. Show that $\Delta BCD \sim \Delta BAC$.
6. If $AD = 7$ and $DB = 9$, what is BC?
7. If $AD = x$ and $AB = c$, show that $AC = \sqrt{xc}$.
8. If $BD = y$ and $AB = c$, then $BC =$ __?__
9. Show that $(AC)^2 + (BC)^2 = (AB)^2$.

MEAN PROPORTIONAL

Your work in Part H leads to two important theorems which we shall state after introducing a new term.



Since ADC $\longleftrightarrow$ CDB is a similarity, it follows from the definition of similarity that

$$\frac{AD}{CD} = \frac{DC}{DB}.$$

That is, it follows that $\frac{x}{h} = \frac{h}{y}$.

So, $h^2 = xy$, and $h = \sqrt{xy}$.

The segment $\overline{CD}$ is said to be a <u>mean proportional</u> between the segments $\overline{AD}$ and $\overline{DB}$. You can compute the measure of $\overline{CD}$ just by computing the square root of the product of the measures of $\overline{AD}$ and $\overline{DB}$. So, we have the following theorem:

> <u>Theorem 7-4.</u>
>
> The altitude to the hypotenuse of a right triangle is a mean proportional between the segments into which the foot of the altitude divides the hypotenuse.

Also, since ACD $\longleftrightarrow$ ABC is a similarity, it follows that

$$\frac{AD}{AC} = \frac{AC}{AB}.$$

So, $\overline{AC}$ is a mean proportional between $\overline{AD}$ and $\overline{AB}$. Similarly, $\overline{BC}$ is a mean proportional between $\overline{BD}$ and $\overline{BA}$. These results are useful in establishing one of the most important theorems in geometry, since from them it follows that

$$b^2 = xc,$$

and

$$a^2 = yc.$$

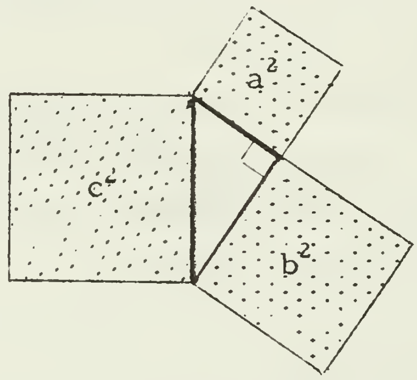
Therefore,

$$\begin{aligned} a^2 + b^2 &= yc + xc \\ &= (y + x)c \\ &= c^2. \end{aligned}$$

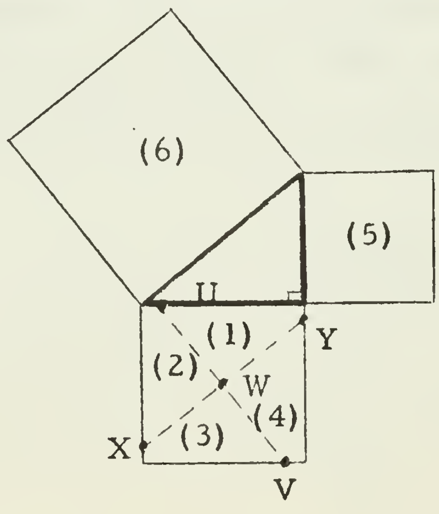
Thus, we have proved the Pythagorean Theorem:

> Theorem 7-5.
>
> The sum of the squares of the measures of the legs of a right triangle is the square of the measure of the hypotenuse.

The Pythagorean Theorem is a surprising result, especially if you interpret it in terms of area. Make a careful drawing of a right triangle. Then, draw squares on each of its three sides. Since the area-measure of a square is the square of the measure of one of its

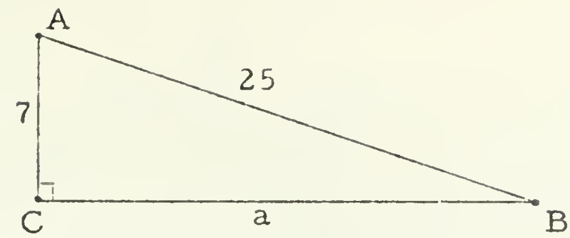


sides, the Pythagorean Theorem tells us that the sum of the area-measures of the squares on the legs is the area-measure of the square on the hypotenuse. In fact, using theorems on area-measure which we have not yet proved, we can give other proofs of the Pythagorean Theorem by showing how the two smaller square regions can be cut up and their pieces fitted together to form the larger square region. For example, it is possible to prove that the square region (6) can be dissected into five regions, four of which are congruent to (1), (2), (3), and (4), and the fifth congruent to (5).



W is the point of intersection of the diagonals of the square; $\overleftrightarrow{XY}$ is parallel to the hypotenuse of the right triangle; $\overleftrightarrow{UV}$ is perpendicular to $\overleftrightarrow{XY}$

Example 1. Find BC.



Since $a^2 + 7^2 = 25^2$,

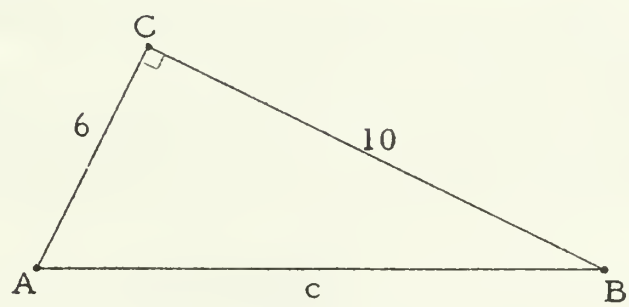
it follows that $a^2 + 49 = 625$,

and that $a^2 = 576$.

Hence, $a = \sqrt{576} = 24$.

So, BC = 24.

Example 2. Find AB.



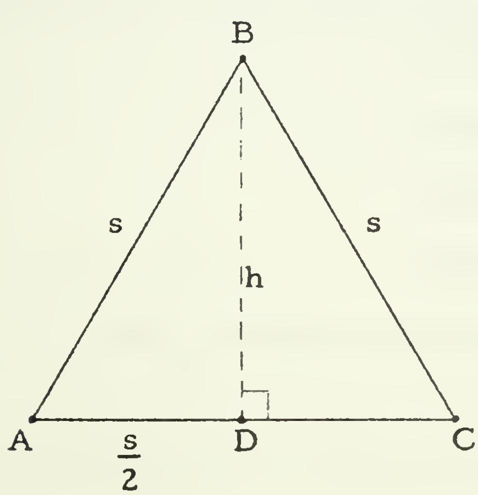
Since $c^2 = 6^2 + 10^2$,

it follows that $c^2 = 136$,

and that $c = \sqrt{136} = \sqrt{4 \cdot 34} = 2\sqrt{34}$.

So, $AB = 2\sqrt{34}$.

Example 3. Find the measure h of an altitude of an equilateral triangle each of whose sides has measure s.



$$s^2 = h^2 + \left(\frac{s}{2}\right)^2$$

$$s^2 = h^2 + \frac{s^2}{4}$$

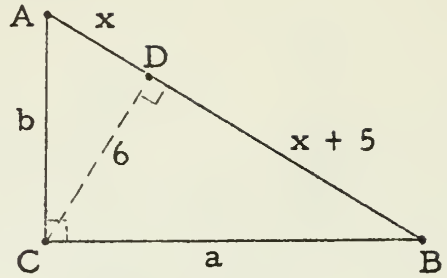
$$4s^2 = 4h^2 + s^2$$

$$4h^2 = 3s^2$$

$$h^2 = \frac{3s^2}{4}$$

$$h = \sqrt{\frac{3s^2}{4}} = \sqrt{\frac{s^2}{4} \cdot 3} = \frac{s\sqrt{3}}{2}$$

Example 4. Find AC and CB.



Since $\overleftrightarrow{CD}$ is the altitude to the hypotenuse of ΔACB, it follows that

$$6^2 = x(x + 5).$$

So,

$$x^2 + 5x - 36 = 0,$$
$$(x + 9)(x - 4) = 0,$$
$$x + 9 = 0 \text{ or } x - 4 = 0.$$

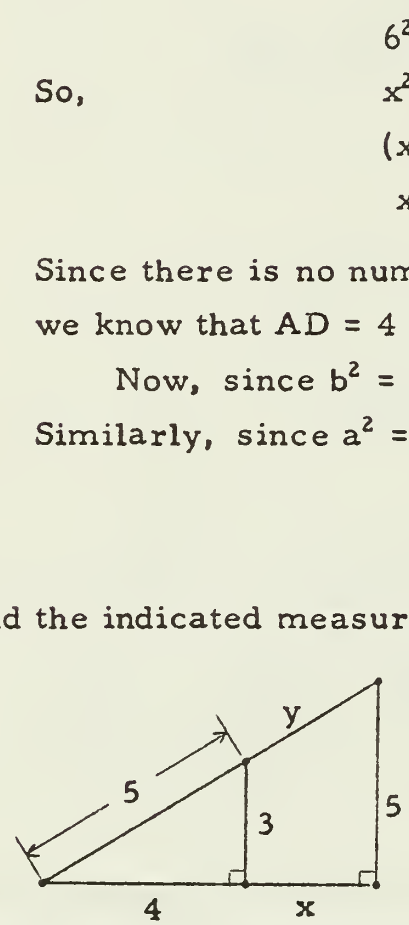
Since there is no number of arithmetic which satisfies '$x + 9 = 0$', we know that AD = 4 and BD = 9.

Now, since $b^2 = 4^2 + 6^2$, it follows that $AC = \sqrt{52} = 2\sqrt{13}$. Similarly, since $a^2 = 9^2 + 6^2$, it follows that $BC = \sqrt{117} = 3\sqrt{13}$.

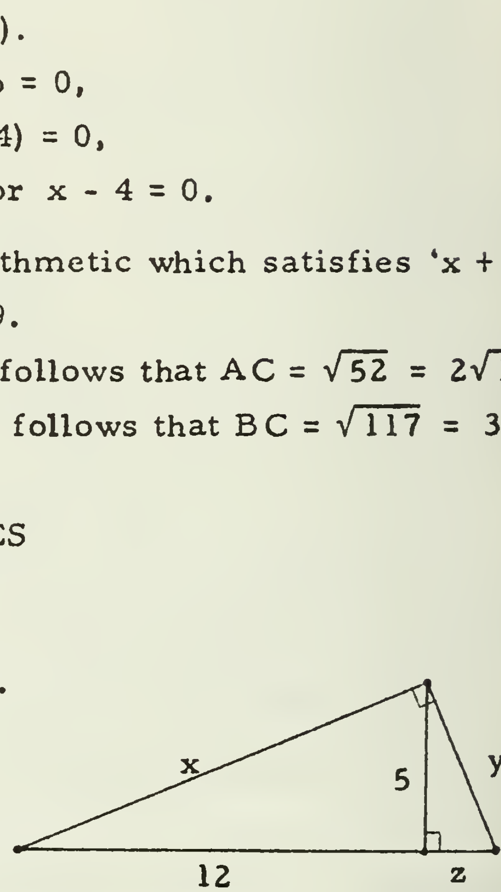
EXERCISES

A. Find the indicated measures.

1.



2.



B. Suppose that ΔABC is an isosceles triangle with $\angle C$ a right angle.

1. If AC = 10, what are BC and AB?
2. If AC = 7, what are BC and AB?
3. If AB = 10, what are AC and BC?
4. If AB = 9, what are AC and BC?
5. Find the measure h of the hypotenuse if the measure of each leg is s.
6. Find the measure s of each leg if the measure of the hypotenuse is h.

C. The following table refers to a right triangle ΔMNP with right angle $\angle P$. Complete the table.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
MN		13		5	$\sqrt{26}$			2
NP	3	5	9	2		15	7	4
PM	4		40		$\sqrt{2}$	35	14	

D. Suppose that in ΔABC, $m(\angle A) = 30$ and $m(\angle B) = 60$.

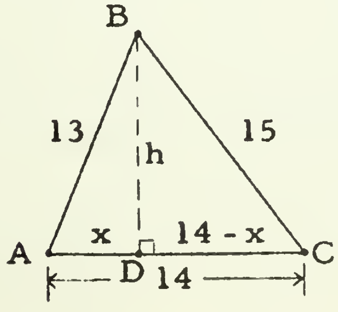
1. If $BC = 8$, what are AB and AC?
2. If $BC = 85$, what are AB and AC?
3. If $BC = x$, what are AB and AC?
4. If $AB = 100$, what are BC and AC?
5. If $AB = 9$, what are BC and AC?
6. If $AB = x$, what are BC and AC?
7. If $AC = 10\sqrt{3}$, what are BC and AB?
8. If $AC = 10$, what are BC and AB?
9. If $AC = x$, what are BC and AB?

E. 1. If $AB = 7$ and $BC = 9$, what is the product of the measures of the diagonals of the rectangle ABCD?

2. What is the measure of a diagonal of a square whose perimeter is 100?

3. A median of an equilateral triangle is 24 inches long. What is the number of inches in a side?

4. If ABCD is a rhombus and $AC = 6$ and $BD = 4$, find AB.

5. 

Given: $AB = 13$, $BC = 15$, $AC = 14$, $\overleftrightarrow{BD} \perp \overleftrightarrow{AC}$

Find: AD, DC, BD

6. Quadrilateral ABCD is a parallelogram with AB = 10, BC = 9, and AC = 17, and $\angle DAB$ is an acute angle. Find the distance between $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$. [Recall that the distance between two parallel lines is the measure of a segment which is perpendicular to the lines and whose end points are on the lines.]

7. In ΔABC, AB = 20, AC = 28, and $\angle A$ is an angle of 30°. What is the measure of the altitude of ΔABC from the vertex B?

8. If a leg of an isosceles triangle is 16 inches long, and one of its base angles is an angle of 30°, how long is the base?

9. If one angle of a parallelogram is an angle of 60°, and the measures of two adjacent sides are 6 and 4, what are the distances between the pairs of parallel sides?

10. The altitude $\overline{BD}$ is also the angle bisector of ΔABC from B. If $m(\angle A) = 2 \cdot m(\angle ABD)$, and AD = 10, what are BD and BC?

11. Quadrilateral ABCD is a trapezoid with $\overline{BC} \parallel \overline{AD}$. If $\angle A$ is an angle of 45°, and AB = 6, what is the distance between the parallel sides?

12. If $m(\angle A) = 45$, AB = 6, and AC = 20, what is BC?

13. (a) If AB = 6, CD = 24, and $\overline{EF}$ is a mean proportional between $\overline{AB}$ and $\overline{CD}$, what is EF?

    (b) If the measure of a side of a square is s, what is the measure of a mean proportional between any two of its sides?

    (c) Show that the median to the hypotenuse of a right triangle is a mean proportional between the segments into which the median divides the hypotenuse.

F. 1. The following table refers to right triangles whose side measures are whole numbers, and whose smallest side measure is an odd number. Complete the table.

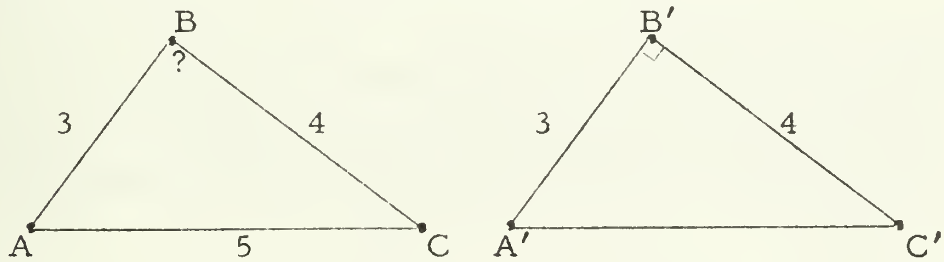
a	3	5	7	9	11	13	15	17	19	21	23	25
b	4	12	24	40								
c	5	13	25		61							

2. If the smallest side measure of a right triangle like those in Exercise 1 is $2k + 1$, what are the measures of the longer leg and the hypotenuse?

3. Show that if the measures of a leg and the hypotenuse of a right triangle are x and $x + 1$, respectively, then the measure of the other leg is $\sqrt{2x + 1}$.

* * *

You know that if the measures of the legs of a right triangle are 3 and 4, respectively, then the measure of the hypotenuse is 5. But, given a triangle whose sides are 3, 4, and 5 units long, respectively, can you conclude that the triangle is a right triangle?



We want to show that ΔABC is a right triangle. Consider the right triangle, $\Delta A'B'C'$ with right angle at $\angle B$, and legs of measure 3 and 4, respectively. What is $A'C'$? [How do you know this?]

Are the triangles, ΔABC and $\Delta A'B'C'$, congruent? Describe a matching of their vertices which is a congruence. What theorem tells you that this matching is a congruence? Are $\angle B$ and $\angle B'$ corresponding angles with respect to this matching? Can you conclude from this that ΔABC is a right triangle?

The foregoing argument for a triangle with side measures 3, 4, and 5 can be easily generalized. Given a triangle, ΔMNP, such that $(MN)^2 + (NP)^2 = (MP)^2$. ... [Finish the proof.] So, we have the following:

> <u>Theorem 7-6.</u>
>
> If the sum of the squares of the measures of two sides of a triangle is the square of the measure of the third side, then the triangle is a right triangle with the third side as hypotenuse.

* * *

G. Which of the following triples are measures of the sides of a right triangle?

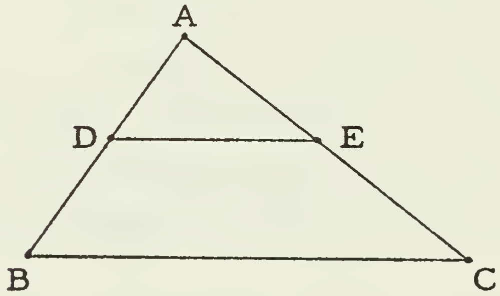
1. $(6, 8, 10)$ 2. $(5, 5, 5\sqrt{2})$ 3. $(\sqrt{7}, \sqrt{10}, \sqrt{70})$
4. $(8, 15, 17)$ 5. $(0.5, 1.2, 1.3)$ 6. $(1, \sqrt{2}, \sqrt{3})$
7. $(2, 2\sqrt{3}, 4)$ 8. $(1, \frac{4}{3}, \frac{5}{3})$ 9. $(9, 10, 20)$

H. You know that if the measures of the sides of a triangle are 3, 4, and 5 then the triangle is a right triangle. Also, 6, 8, and 10 are measures of the sides of a right triangle. Show that, for each non-zero number k of arithmetic, 3k, 4k, and 5k are measures of the sides of a right triangle.

☆ I. 1. Suppose p and q are whole numbers such that $p > q$. Show that if $p^2 - q^2$ and $2pq$ are measures of two sides of a triangle and $p^2 + q^2$ is the measure of the third then the triangle is a right triangle.

2. Use the expressions in Exercise 1 to obtain ten triples of whole numbers which are measures of the sides of right triangles.

J. 1.



Hypothesis: $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$

Conclusion: $\triangle ADE \sim \triangle ABC$

2. Prove:

> **Theorem 7-7.**
> Congruent triangles are similar.

3. Prove:

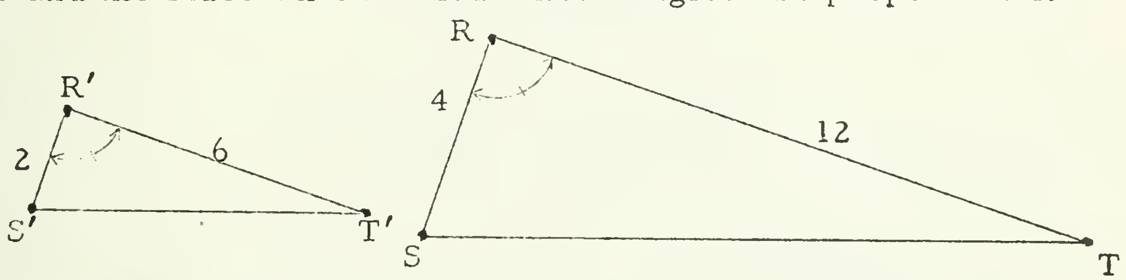
> **Theorem 7-8.**
> If two triangles are similar to a third, they are similar to each other.

MORE TRIANGLE-SIMILARITY THEOREMS

By definition, a necessary and sufficient condition that two triangles be similar is that there be a matching of their vertices such that corresponding angles are congruent and corresponding sides are proportional. Now, just as in the case of triangle-congruence, there are sufficient conditions for triangle-similarity which are simpler to apply than the one given in the definition. For example, we have already shown that a sufficient condition for the similarity of two triangles is that two angles of one be congruent, respectively, to two angles of the other. [Is this condition also necessary?]

Now, let's investigate other possibilities for sufficient conditions for triangle-similarity.

By analogy with the s.a.s. theorem for congruences, you might guess that two triangles are similar if they agree in one pair of their angles and the sides which include these angles are proportional.



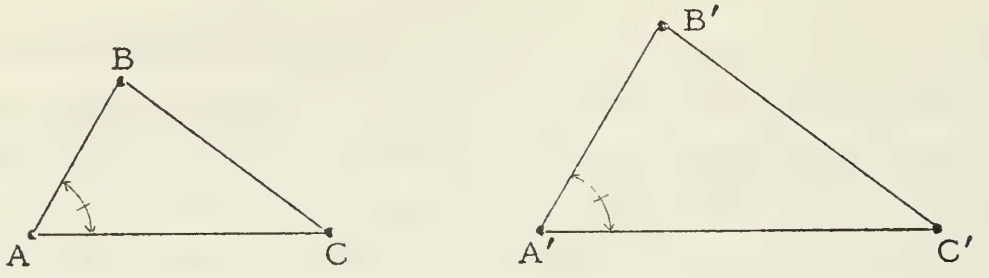
Are these triangles similar?

It is easy to show that they are. Let M and N be the midpoints of $\overleftrightarrow{RS}$ and $\overleftrightarrow{RT}$, respectively. What do you notice about $\Delta R'S'T'$ and ΔRMN? What do you notice about ΔRST and ΔRMN? Do you see how to use Theorems 7-7 and 7-8 to show that $\Delta R'S'T' \sim \Delta RST$?

Let us prove the following theorem:

> <u>Theorem 7-9.</u> [s.a.s. similarity theorem]
>
> If, for some matching of the vertices of one triangle with those of a second, one angle of the first is congruent to the corresponding angle of the second and the sides including the first angle are proportional to the corresponding sides of the second triangle, then the matching is a similarity.

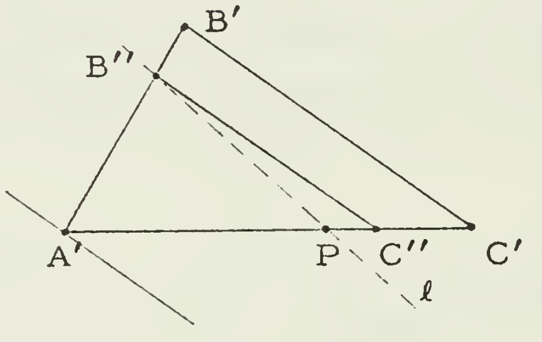
We shall do so by solving the following problem.



<u>Hypothesis:</u> $\angle A \cong \angle A'$, $\frac{A'B'}{AB} = \frac{A'C'}{AC}$

<u>Conclusion:</u> $ABC \longleftrightarrow A'B'C'$ is a similarity

Let B'' and C'' be the points on $\overrightarrow{A'B'}$ and $\overrightarrow{A'C'}$, respectively, such that $A'B'' = AB$ and $A'C'' = AC$. We shall now prove that



$\Delta A'B''C'' \sim \Delta A'B'C'$. [Since $\Delta ABC \cong \Delta A'B''C''$, it will follow that $\Delta ABC \sim \Delta A'B'C'$. Explain.] To do so, it will be enough to show that $\angle B' \cong \angle A'B''C''$. This will follow if we show that $\overleftrightarrow{B''C''} \parallel \overleftrightarrow{B'C'}$. [Why?]

Let ℓ be the parallel to $\overleftrightarrow{B'C'}$ through B'', and let P be the point in which ℓ intersects $\overrightarrow{A'C'}$.

Now, by Theorem 7-1, it follows that

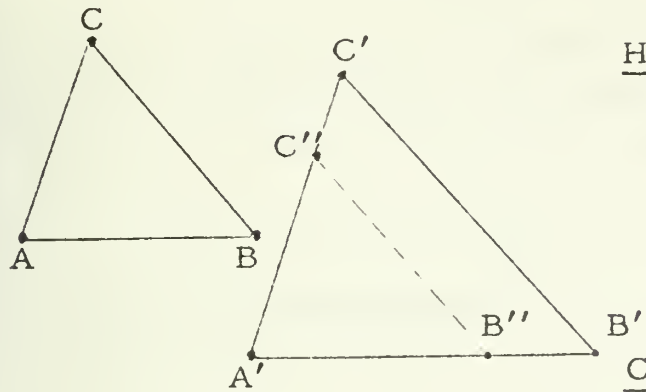
$$(1) \quad \frac{A'B'}{A'B''} = \frac{A'C'}{A'P}.$$

But, by hypothesis and the fact that $AB = A'B''$ and $AC = A'C''$, we also know that

$$(2) \quad \frac{A'B'}{A'B''} = \frac{A'C'}{A'C''}.$$

So, by algebra, we derive from (1) and (2) that $A'P = A'C''$. Since P and C'' both belong to $\overrightarrow{A'C'}$, it follows that $P = C''$. Hence, $\overleftrightarrow{B''C''} \parallel \overleftrightarrow{B'C'}$. Thus, we have proved Theorem 7-9.

Now, we have two basic triangle-similarity theorems--a.a. and s.a.s. Let's try for s.s.s.



<u>Hypothesis:</u> $\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA}$

<u>Conclusion:</u> $A'B'C' \leftrightarrow ABC$ is a similarity

Let B'' be the point in $\overrightarrow{A'B'}$ such that $A'B'' = AB$, and let C'' be the point in $\overrightarrow{A'C'}$ such that $A'C'' = AC$. Now, by hypothesis,

$$\frac{A'B'}{AB} = \frac{C'A'}{CA}.$$

So,

$$\frac{A'B'}{A'B''} = \frac{C'A'}{C''A'}. \qquad \text{[Explain.]}$$

Also, $\angle A' \cong \angle A'$. Therefore, by the s.a.s. similarity theorem,

$$(*) \qquad A'B'C' \leftrightarrow A'B''C'' \text{ is a similarity.}$$

By the definition of similarity,

$$(1) \qquad \frac{B'C'}{B''C''} = \frac{A'B'}{A'B''}.$$

But, by hypothesis and the fact that $A'B'' = AB$, we know that:

$$(2) \qquad \frac{B'C'}{BC} = \frac{A'B'}{A'B''}$$

So, (1) and (2) tell us that $B''C'' = BC$.

Now, the s.s.s. congruence theorem tells us that

$$A'B''C'' \leftrightarrow ABC \text{ is a congruence,}$$

and, so, by Theorem 7-7,

$$(**) \qquad A'B''C'' \leftrightarrow ABC \text{ is a similarity.}$$

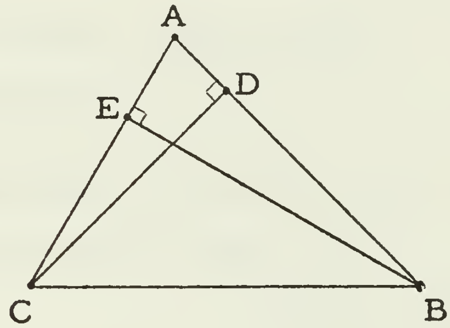
By Theorem 7-8, it follows from (*) and (**) that

$$A'B'C' \leftrightarrow ABC \text{ is a similarity.}$$

Thus, we have proved the following theorem:

> <u>Theorem 7-10.</u> [s.s.s. similarity theorem]
> If, for some matching of the vertices of one triangle with those of a second, corresponding sides are proportional, then the matching is a similarity.

<u>Example 1.</u>



<u>Hypothesis:</u> $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{BE} \perp \overleftrightarrow{AC}$

<u>Conclusion:</u> $CD \cdot AB = BE \cdot AC$

<u>Plan.</u> The conclusion will follow if we can show that $\frac{CD}{BE} = \frac{AC}{AB}$. This will follow if we can show that $CDA \longleftrightarrow BEA$ is a similarity. And, this is easy to do because $\angle A \cong \angle A$, and $\angle D$ and $\angle E$ are right angles.

<u>Solution.</u>

Consider the triangles $\triangle CDA$ and $\triangle BEA$. Since perpendicular lines contain right angles and all right angles are congruent, it follows that $\angle CDA \cong \angle BEA$. Also, $\angle A \cong \angle A$. So, by the a.a. similarity theorem, $CDA \longleftrightarrow BEA$ is a similarity. By the definition of similarity, it follows that

$$\frac{CD}{BE} = \frac{AC}{AB}.$$

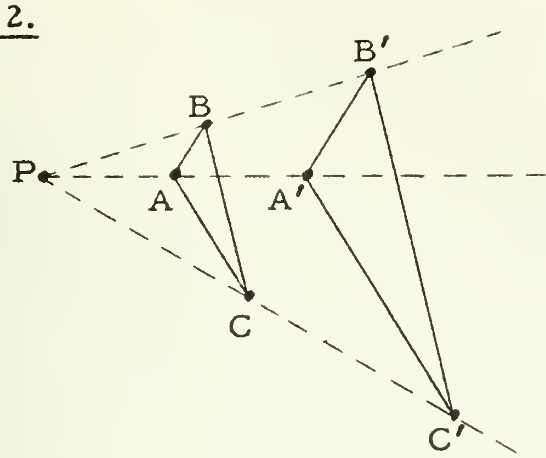
Therefore,

$$DC \cdot AB = EB \cdot AC.$$

<u>Query.</u> Notice that $\overleftrightarrow{CD}$ and $\overleftrightarrow{BE}$ are altitudes of $\triangle ABC$, and that $\angle A$ is an acute angle. Does the conclusion still follow if $\angle A$ is a right angle? If $\angle A$ is an obtuse angle?

<u>Note.</u> Use the result of this example to compute the measure of the altitude to the hypotenuse of a right triangle whose legs measure 3 and 4, respectively.

Example 2.



Hypothesis: $PA' = 2 \cdot PA$,
$PB' = 2 \cdot PB$,
$PC' = 2 \cdot PC$

Conclusion: $\Delta ABC \sim \Delta A'B'C'$

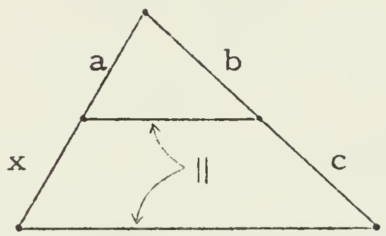
Solution.

(1)	A′, P, B′, and A, P, B are vertices of triangles	[figure]
(2)	$\angle APB \cong \angle A'PB'$	[Identity; def. of cong. angles]
(3)	$\frac{A'P}{AP} = \frac{PB'}{PB}$	[Hypothesis]
(4)	s.a.s. similarity theorem	[theorem]
(5)	$A'PB' \longleftrightarrow APB$ is a similarity	[(1), (2), (3), and (4)]
(6)	$\frac{A'B'}{AB} = \frac{A'P}{AP}$	[(5); def. of similarity]
(7)	$A'P = 2 \cdot AP$	[Hypothesis]
(8)	$A'B' = 2 \cdot AB$	[(6) and (7)]
(9)	$B'C' = 2 \cdot BC$	[steps like (1) - (8)]
(10)	$C'A' = 2 \cdot CA$	[steps like (1) - (8)]
(11)	A′, B′, C′ and A, B, C are vertices of triangles	[figure]
(12)	s.s.s. similarity theorem	[theorem]
(13)	$A'B'C' \longleftrightarrow ABC$ is a similarity	[(11), (8), (9), (10), and (12)]
(14)	$\Delta A'B'C' \sim \Delta ABC$	[(13); def. of similar triangles]

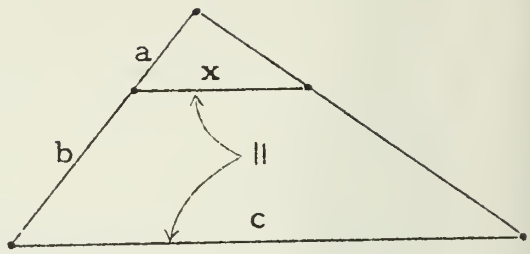
EXERCISES

A. Use the other indicated measures to compute x.

1.



2.



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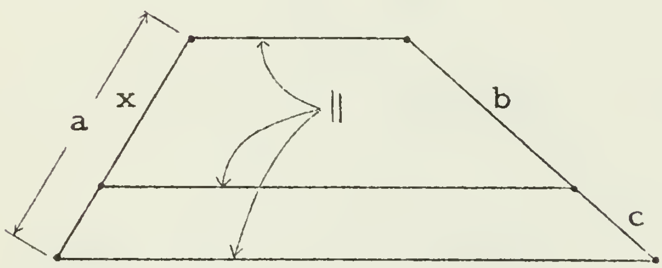

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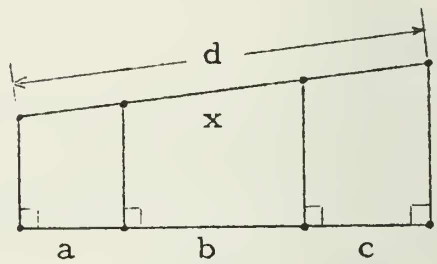
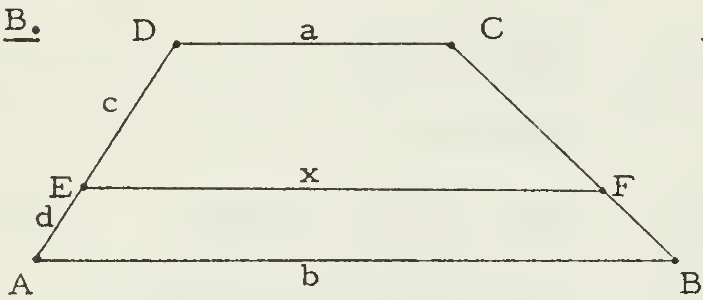


☆ B.

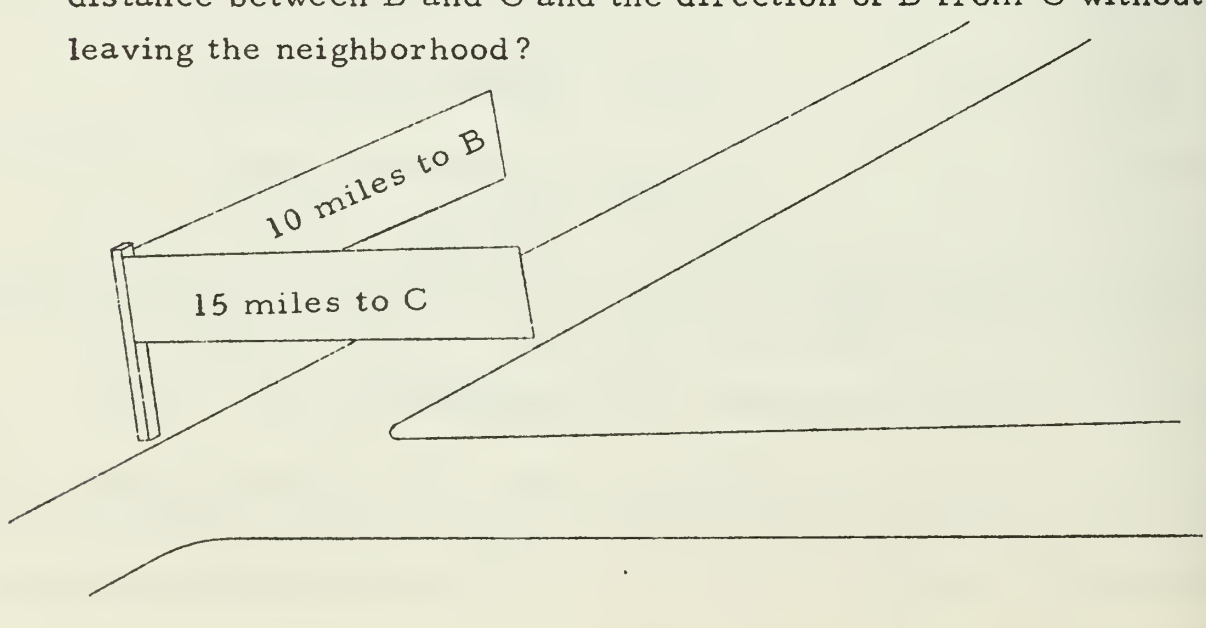


Given: ABCD is a trapezoid with $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, $\overleftrightarrow{EF} \parallel \overleftrightarrow{AB}$

Use a, b, c, and d to compute x.

[Hint. Consider $\triangle PAB$ where $\{P\} = \overleftrightarrow{AD} \cap \overleftrightarrow{BC}$.]

C. Suppose that you are at a crossroads. How could you find the distance between B and C and the direction of B from C without leaving the neighborhood?



D. 1. Suppose that $\overline{AB} \parallel \overline{CD}$ and that $\overline{AC} \cap \overline{BD} = \{E\}$. If $AB = 7 \cdot CD$, show that $BD = 8 \cdot ED$.

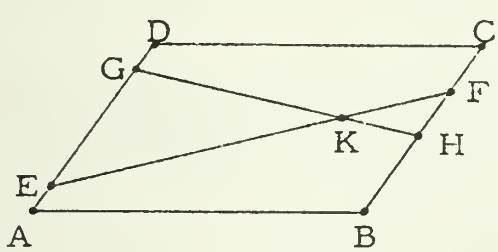
2. Given that $AC = 3$, $AB = 5$, $CB = 7$, and that $ED = 21$, $EF = 15$ and $FD = 9$. Is there a similarity-matching of the vertices of $\triangle ABC$ with those of $\triangle DEF$? If there is, give it.

3. What is the measure of the altitude to the longest side of a triangle whose side measures are 36, 48, and 60?

4. Suppose that $ABC \longleftrightarrow A'B'C'$ is a similarity. If $\overline{AM}$ and $\overline{A'M'}$ are the medians from A and A', respectively, of $\triangle ABC$ and $\triangle A'B'C'$, show that $AM/A'M' = AB/A'B'$.

E. 1. [Refer to the figure for Example 2 on page 6-215.] Given that $\overline{AB} \parallel \overline{A'B'}$ and $\overline{AC} \parallel \overline{A'C'}$. Show that $ABC \longleftrightarrow A'B'C'$ is a similarity, and that $\overleftrightarrow{BC} \parallel \overleftrightarrow{B'C'}$.

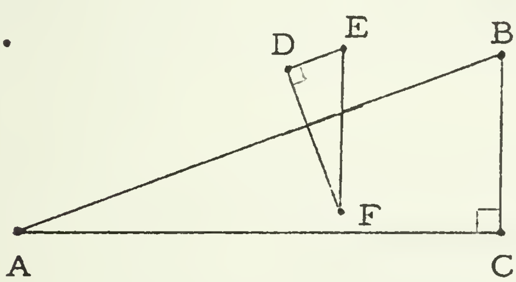
2.



Hypothesis: ABCD is a parallelogram, $\overline{EF} \cap \overline{GH} = \{K\}$

Conclusion: $GK \cdot KF = EK \cdot KH$

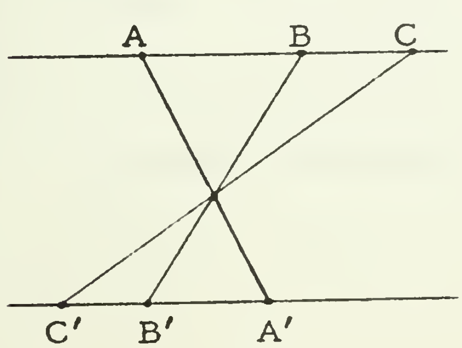
3.



Hypothesis: $\angle C$ and $\angle D$ are right angles, $\overline{ED} \parallel \overline{AB}$, $\overline{EF} \parallel \overline{BC}$

Conclusion: $ED = EF \cdot \dfrac{BC}{BA}$, $DF = EF \cdot \dfrac{CA}{BA}$

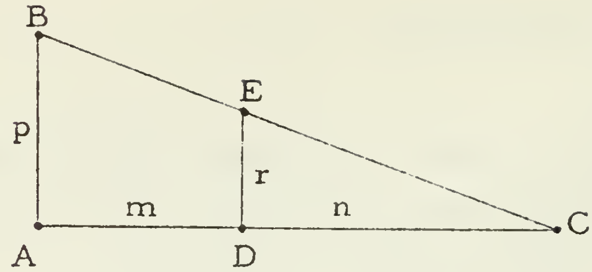
4.



Hypothesis: $\overleftrightarrow{AC} \parallel \overleftrightarrow{C'A'}$

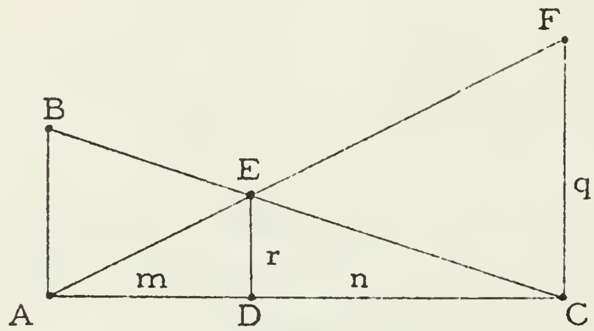
Conclusion: $\dfrac{AB}{AC} = \dfrac{A'B'}{A'C'}$

☆ F. 1.



Suppose that $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$. Use m and n to compute r/p.

2.



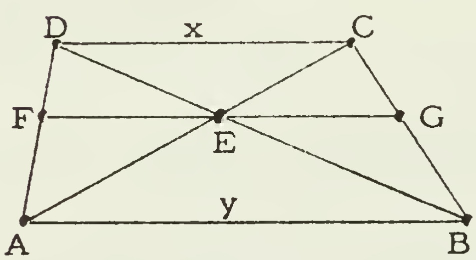
Suppose that $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE} \parallel \overleftrightarrow{CF}$. Use m and n to compute r/q.

3. Use the results of Exercise 1 and 2 to show that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

[Hint. Show first that $\frac{r}{p} + \frac{r}{q} = 1$.]

4. You can estimate the root of the equation '$\frac{1}{3} + \frac{1}{5} = \frac{1}{x}$' by making a careful drawing like the figure in Exercise 2. Do so.

5.



ABCD is a trapezoid with bases $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. The segment $\overleftrightarrow{FG}$ is parallel to the bases and contains the point of intersection of the diagonals of the trapezoid. Use x and y to compute FG.

[Hint. Use Exercise 3.]

SUMMARY OF SECTION 6.07

Notation and Terminology

are proportional to	[6-188]	necessary condition	[6-180]
in proportion	[6-187]	ratio of two segments	[6-187]
mean proportional	[6-203]	similarity	[6-192]
necessary and sufficient condition	[6-182]	similar polygon	[6-192]
		sufficient condition	[6-180]

p : q [6-186] $\triangle ABC \sim \triangle DEF$ [6-198]

Theorems

7-1. Three parallel lines cut off proportional segments on each two transversals.

7-2. If, for some matching of the vertices of one triangle with those of a second, each of two angles of the first triangle is congruent to the corresponding angle of the second, the matching is a similarity. [a.a. similarity theorem]

7-3. If two right triangles agree in a pair of acute angles, the triangles are similar.

7-4. The altitude to the hypotenuse of a right triangle is a mean proportional between the segments into which the foot of the altitude divides the hypotenuse.

7-5. The sum of the squares of the measures of the legs of a right triangle is the square of the measure of the hypotenuse.

7-6. If the sum of the squares of the measures of two sides of a triangle is the square of the measure of the third side, then the triangle is a right triangle with the third side as hypotenuse.

7-7. Congruent triangles are similar.

7-8. If two triangles are similar to a third, they are similar to each other.

7-9. If, for some matching of the vertices of one triangle with those of a second, one angle of the first is congruent to the corresponding angle of the second and the sides including the first angle are proportional to the corresponding sides of the second triangle, then the matching is a similarity.

7-10. If, for some matching of the vertices of one triangle with those of a second, corresponding sides are proportional, then the matching is a similarity.

[Supplementary exercises are on page 6-433.]

EXPLORATION EXERCISES

A. Use the Pythagorean Theorem or what you know about 45-45-90 triangles to derive a formula for computing the measure d of a diagonal of a square from its side-measure s.

B. 1. Derive a formula for computing the measure d_1 of one of the shorter diagonals of a regular hexagon from its side-measure s.

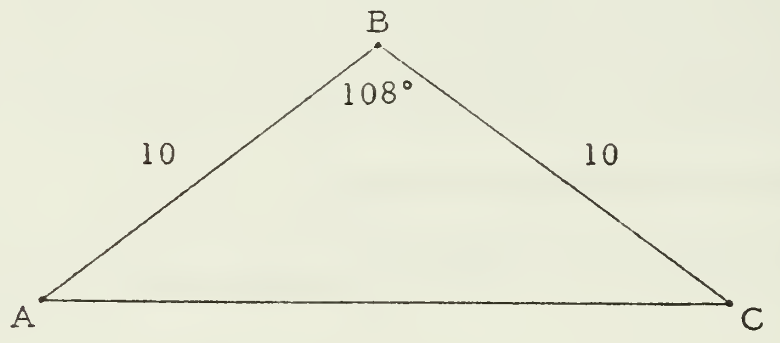
2. Repeat for the longer diagonals.

C. Try to derive a formula like those in Parts A and B for a regular pentagon.

☆ D. Repeat Part C for a regular octagon.

*

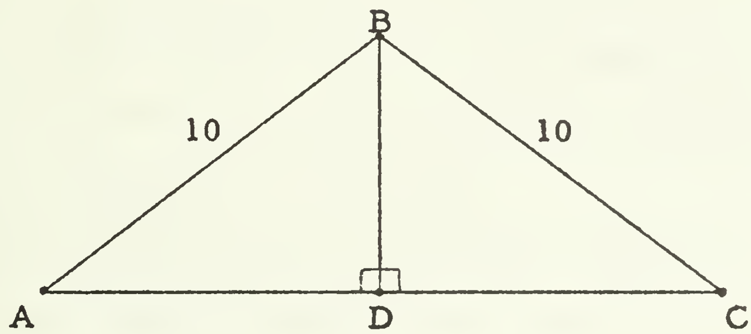
6.08 Trigonometric ratios.--Here is a figure showing two adjacent sides and a diagonal of a regular pentagon:



Given that the side-measure is 10, how can we find AC? Since the pentagon is regular, we know that ∠B is an angle of 108°. [Explain.] So, our job amounts to finding the measure of the base of an isosceles triangle whose legs are 10 units long and whose vertex angle is an angle of 108°.

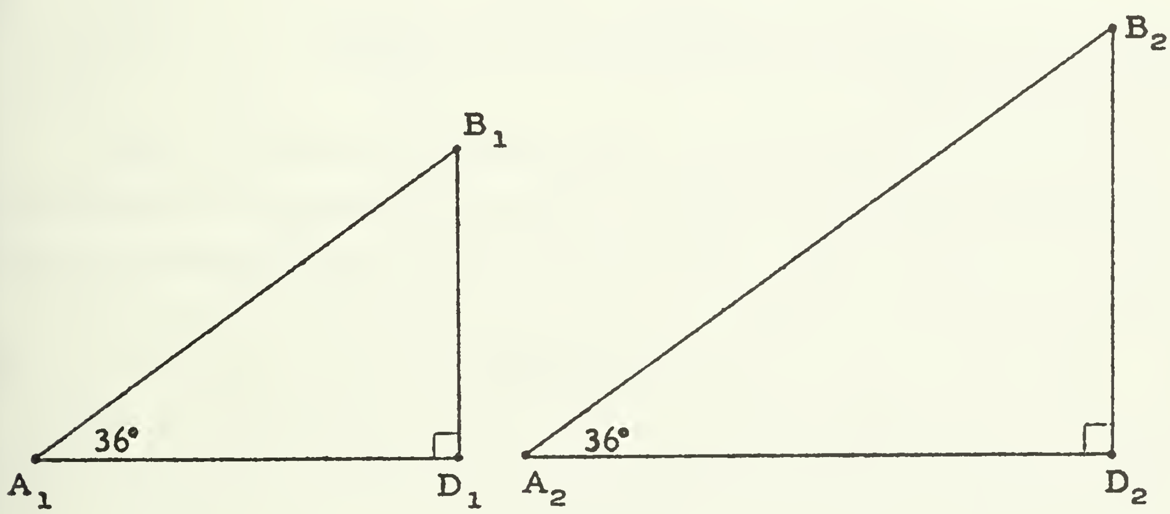
You could get an approximation to AC "good enough for practical purposes" by measuring a scale drawing. Of course, the accuracy of the approximation would be limited because there is a limit to the accuracy with which measuring devices such as rulers and protractors can be made and used.

The problem of finding AC, and many other problems, can be solved by using a method which does not involve the inaccuracies of scale drawings. The method consists in reducing the problem to simpler problems about <u>right</u> triangles. In the case at hand, consider the two right triangles ΔABD and ΔCBD, where D is the foot of the altitude from B.



Clearly, $AC = AD + DC$. In fact, since ΔABC is isosceles, $AC = 2 \cdot AD$. So, we can find AC just by finding AD and doubling. Hence, the problem is reduced to finding the measure of the leg "adjacent" to one of the acute angles of a right triangle when the measures of that angle and the hypotenuse are known. [What is the measure of $\angle A$? How do you know?]

Now, let's consider any two right triangles each having an acute angle congruent to $\angle A$. The matching of the vertices



$A_1B_1D_1 \leftrightarrow A_2B_2D_2$ is a similarity. So, $\dfrac{A_1D_1}{A_2D_2} = \dfrac{A_1B_1}{A_2B_2}$. Transforming this proportion gives us:

$$(*) \qquad \frac{A_1D_1}{A_1B_1} = \frac{A_2D_2}{A_2B_2}$$

This last proportion tells us that, for any two 36°-right triangles, the leg adjacent to the 36° angle has the same ratio to the hypotenuse. In the given problem, we are looking for the measure, AD, of the leg adjacent to the 36° angle of a right triangle whose hypotenuse has measure 10. So, since AD/10 is the common ratio referred to in (*), all we need to do to compute AD is to multiply the common ratio by 10.

In later courses you will learn how one can compute approximations to such ratios, with any desired degree of accuracy. Because these ratios are useful in solving a great variety of problems, approximations to them are listed in tables called <u>tables of trigonometric ratios</u>. A portion of such a table is given below. [The complete table is on page 6-231.]

Angle	sin	cos	tan
34°	.5592	.8290	.6745
35°	.5736	.8192	.7002
36°	.5878	.8090	.7265
37°	.6018	.7986	.7536

The table lists approximations for three trigonometric ratios associated with 36°-right triangles. The ratio we want is referred to in the column headed 'cos'. So, the table tells us that AD/10 is 0.809 correct to the nearest 0.0001. So, AD is 8.09 correct to the nearest 0.001. [Of course, you could get better approximations to AD by using tables which listed better approximations to the trigonometric ratios.]

Now, compute the approximation correct to the nearest 0.01 to the measure of a diagonal of a regular pentagon whose side-measure is 10. Repeat for a regular pentagon whose side-measure is 8.

If the measure of a diagonal of a regular pentagon is 162, compute an approximation to the perimeter of the pentagon.

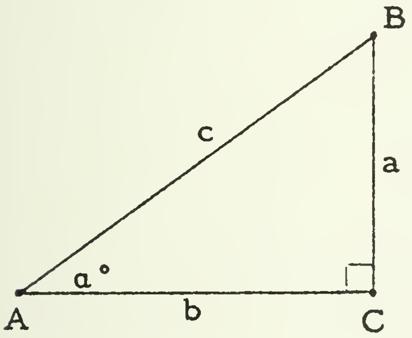
If the hypotenuse $\overleftrightarrow{MT}$ of $\triangle MTS$ is 20 units long, and $\angle T$ is an angle of 35°, compute an approximation to the measure of $\overleftrightarrow{ST}$.

Now, let's consider the other two trigonometric ratios referred to in the table. For a 36°-right triangle, the table tells us that the ratio of the leg opposite the 36° angle to the hypotenuse is [approximately] 0.5878, and that the ratio of the leg opposite the 36° angle to the leg adjacent to the 36° angle is [approximately] 0.7265. The first of these ratios is called <u>the sine ratio</u> for a 36° angle; the other is called <u>the tangent ratio</u> for a 36° angle. And, the one we use in solving the pentagon problem, the ratio of the leg adjacent to the 36° angle to the hypotenuse, is called <u>the cosine ratio</u> for a 36° angle.

In referring to these ratios, we usually just write:

sin 36° [read this as 'sine 36°'],
cos 36° [read this as 'cosine 36°'],
tan 36° [read this as 'tangent 36°'].

In general:



$$\sin a° = \frac{a}{c} = \frac{m(\text{leg opposite } \angle A)}{m(\text{hypotenuse})}$$

$$\cos a° = \frac{b}{c} = \frac{m(\text{leg adjacent to } \angle A)}{m(\text{hypotenuse})}$$

$$\tan a° = \frac{a}{b} = \frac{m(\text{leg opposite } \angle A)}{m(\text{leg adjacent to } \angle A)}$$

[Sometimes, we shall write 'sin ∠A' or 'cos ∠A' or 'tan ∠A' when we are referring to the trigonometric ratios for ∠A.]

EXERCISES

<u>A.</u> Use the table on page 6-231 to find approximations to these ratios.

1. sin 42°	2. tan 80°	3. cos 15°
4. tan 45°	5. cos 45°	6. sin 30°
7. cos 30°	8. sin 60°	9. tan 1°
10. sin 48°	11. cos 42°	12. tan 18°
13. sin 20.5°	14. tan 31.7°	15. cos 60.2°

B. Use the table to approximate the roots of the following equations.

Sample. $\tan x^\circ = 1.2349$

Solution. From the table we see that

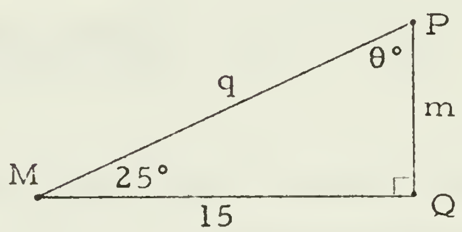
$$\tan 51^\circ = 1.2349, \text{ approximately.}$$

So, the root of the equation is approximately 51.

1. $\sin t^\circ = 0.8988$
2. $\cos y^\circ = 0.2079$
3. $\tan x^\circ = 0.6745$
4. $\sin s^\circ = 0.829$
5. $\cos z^\circ = 0.9511$
6. $\sin y^\circ = 0.9511$
7. $\sin \angle A = 0.4695$ [So, $\angle A$ is an angle of approximately ____°.]
8. $\tan \angle B = 0.9542$
9. $\cos \angle C = 0.3461$
10. $\sin x^\circ = \cos x^\circ$
11. $\cos \alpha^\circ = \sqrt{3}/2$
12. $\sin \alpha^\circ = \cos (2\alpha)^\circ$
13. $\sin \beta^\circ = \cos(90 - \beta)^\circ$

* * *

Example 1. Find the indicated measures.



[Read 'θ' as 'thayta']

Solution.

By definition, $\tan \angle M = \frac{PQ}{MQ}$. So, $\tan 25^\circ = \frac{m}{15}$, and therefore,

$$m = 15 \cdot \tan 25^\circ.$$

The table tells us that tan 25° is approximately 0.4663. [We shall write '$\tan 25^\circ \doteq 0.4663$' as a short way of saying that tan 25° is approximately 0.4663. Read '$\doteq$' as 'is approximately equal to'.]

$$m \doteq 15 \cdot 0.4663 = 6.9945$$

So, PQ is approximately 7.

Now, we could compute q by using the Pythagorean Theorem [How?], but this would involve a lot of computing. Let's use another trigonometric ratio.

Since $\cos \angle M = \frac{MQ}{MP}$, it follows that $\cos 25° = \frac{15}{q}$. So,

$$q = \frac{15}{\cos 25°}$$

$$\doteq \frac{15}{0.9063}$$

$$\doteq 16.55.$$

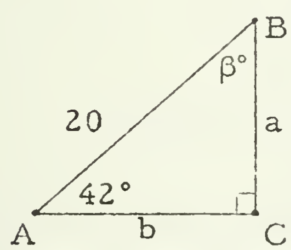
Hence, MP is approximately 17.

Finally, since the acute angles of a right triangle are complementary, $\theta + 25 = 90$. So, $\angle P$ is an angle of 65°.

[Use the Pythagorean Theorem to get an approximate check on your computation.]

C. Find the indicated measures.

1.



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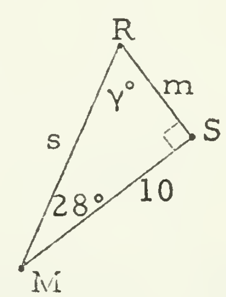

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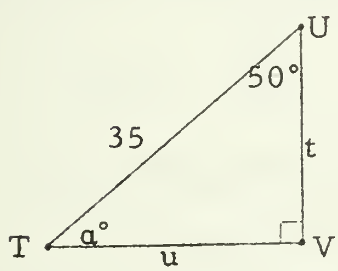

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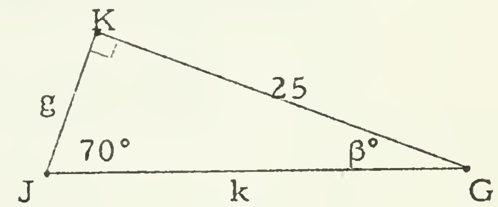

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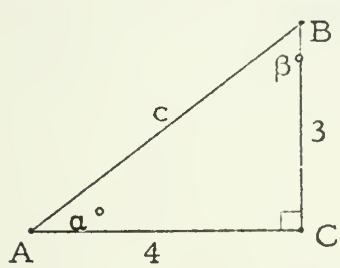

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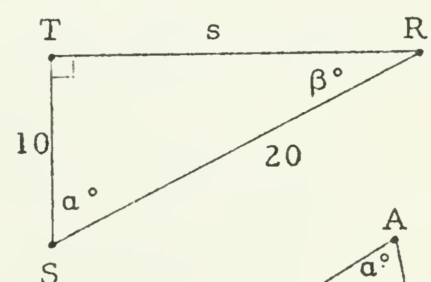

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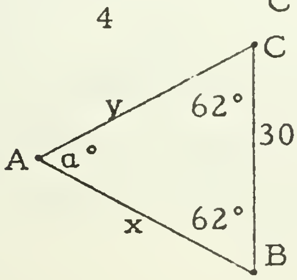

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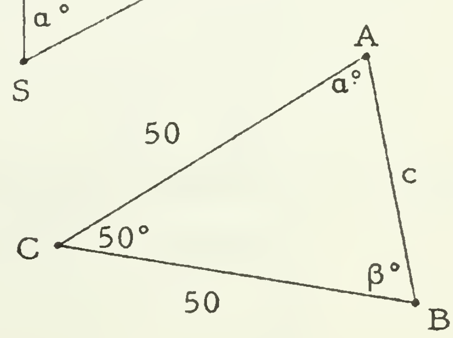
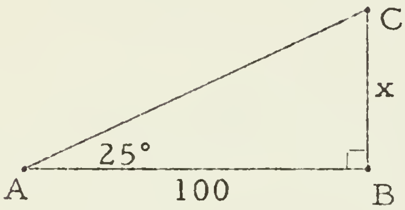


D. Find the indicated measures.

1.



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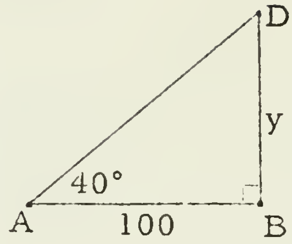

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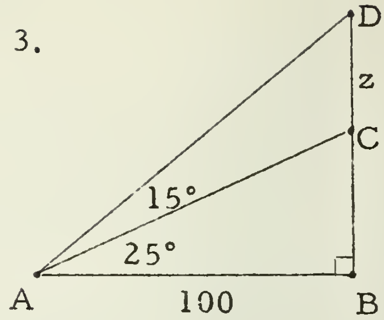

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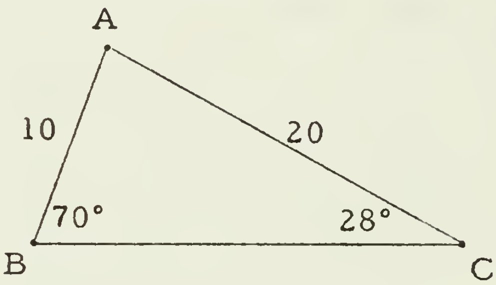
Example 2. Find BC, given the indicated measures.



Solution. Let's find BC by finding BD and DC, where D is the foot of the altitude from A of $\triangle ABC$. [Sketch AD in the figure.]

$$\frac{BD}{10} = \cos 70° \qquad \text{and} \qquad \frac{DC}{20} = \cos 28°$$

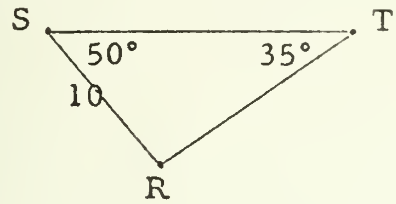
$$\begin{aligned} BD + DC &= 10 \cdot \cos 70° + 20 \cdot \cos 28° \\ &\doteq 10 \cdot 0.342 + 20 \cdot 0.8829 \\ &= 3.42 + 17.658 \\ BC &\doteq 21 \end{aligned}$$

* * *

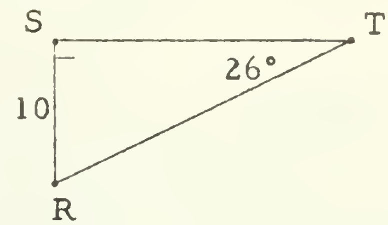
E. 1. Look again at Example 2. From your knowledge of triangle-congruence, you should see that it ought to be possible to find BC by knowing the measures of two angles and just one side. [Explain.] Find BC without using AC.

2. Find ST for each triangle.

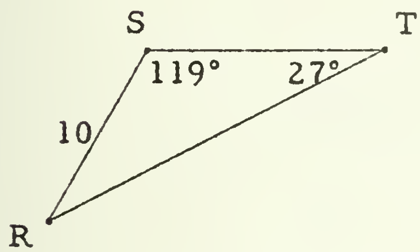
(a)



(b)



(c)



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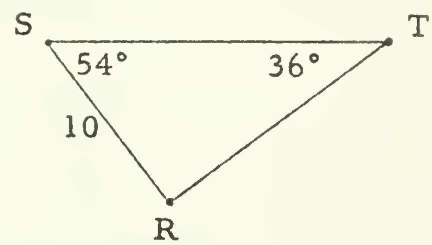
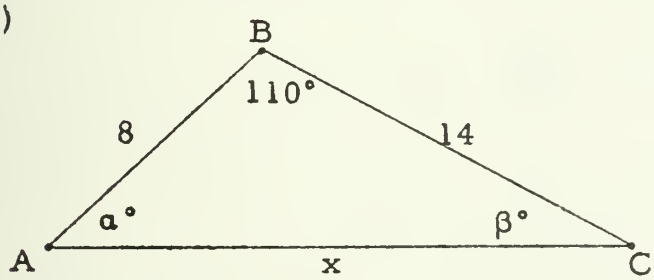


3. Find the indicated measures.

(a)



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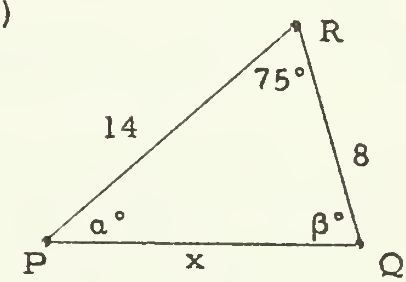
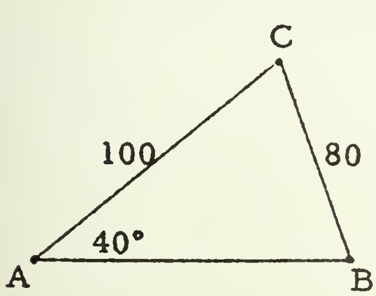
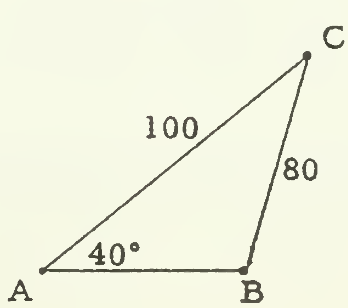


[<u>Hint.</u> Consider the altitude from C of $\triangle ABC$.]

4. If, in $\triangle ABC$, $\angle A$ is an angle of 40°, $AC = 100$, and $CB = 80$, what is AB?

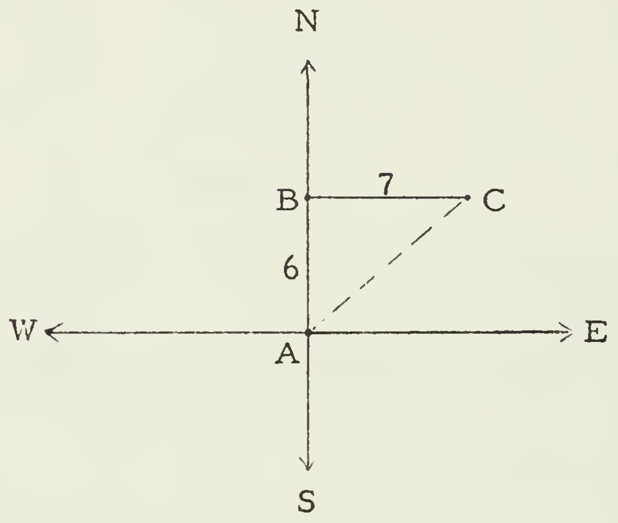




* * *

Example 3. A ship starts from a point A and bears north for 6 miles to B. It then bears east for 7 miles to C. What is the direction of C from A, and how far is C from A?

Solution.



The direction of C from A [that is, the bearing of $\overleftrightarrow{AC}$] is determined by the measure of $\angle BAC$. Since $\triangle ABC$ is right-angled at B,

$$\tan \angle BAC = \frac{BC}{AB} = \frac{7}{6} \doteq 1.17.$$

The table tells us that $\tan 49° \doteq 1.17$. So, the direction of C from A is about N 49° E [read this as 'north 49° east'].

The distance between A and C can be computed by using the Pythagorean Theorem:

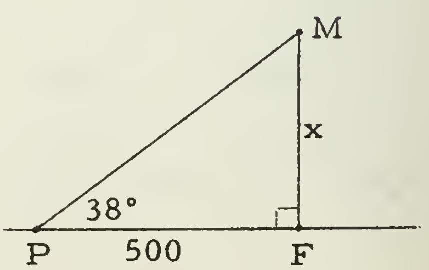
$$AC = \sqrt{6^2 + 7^2} = \sqrt{85} \doteq 9.$$

So, C is about 9 miles from A.

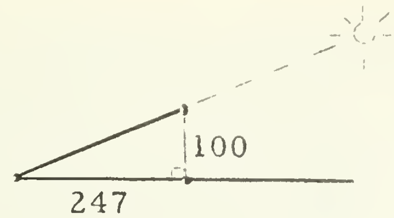
* * *

F. 1. An airplane climbs 1000 feet while traveling 2 miles. Find the "angle of climb".

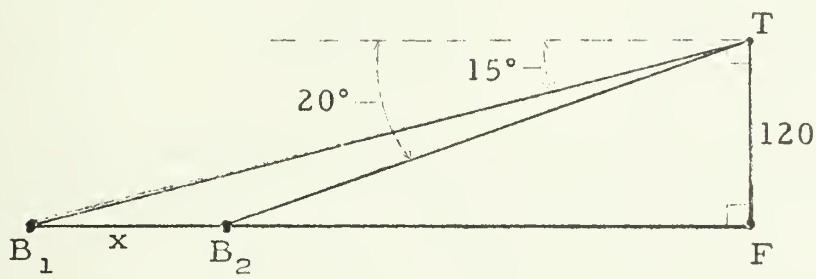
2. The angle of elevation of the top of a monument from a point 500 feet from the base is an angle of 38°. How tall is the monument?



3. Find the measure of the angle of elevation of the sun when a flagpole 100 feet high casts a shadow 247 feet long.



4. The top of a lighthouse is 120 feet above the surface of the water. From the top, the measures of the <u>angles of depression</u> of two buoys are 15° and 20°, respectively. If the buoys are in line with the foot



of the lighthouse, what is the distance between the buoys?

5. Two ships start from the same point at the same time. One ship bears S 20° W and the other bears S 30° E. At the end of an hour, one ship has traveled 20 miles and the other has traveled 30 miles. How far apart are they at that time?

6. A T.V. antenna is mounted on the top of a tall pole. At a point 100 feet from the base of the pole the angle of elevation of the top of the pole is an angle of 20°, and the angle of elevation of the top of the antenna is an angle of 25°. How tall is the antenna?

7. The angle of elevation of a cloud at a point which is 3000 feet from a point which is directly below the cloud is an angle of 57°. How high is the cloud?

8. A highway runs east and west, and is crossed by a dirt road at an angle of 41°. When a car on the highway is 660 feet from the intersection, a second car on the dirt road is due south of the first. If both are heading towards the intersection, the first at 45 miles per hour, the second at 60 miles per hour, which will reach the intersection first?

9. A highway crosses a railroad track at an angle of 60°. A locomotive is 50 yards from the intersection when an automobile is 30 yards from the intersection. What is the distance between the locomotive and the car?

[Supplementary exercises are on page 6-437.]

G. 1. Find the sine and cosine ratios for angles of

5° and 85°,	12° and 78°,	15° and 75°,
25° and 65°,	35° and 55°,	48° and 42°,
60° and 30°,	79° and 11°,	89° and 1°.

Write your results in a table which starts like this:

	5°	85°	12°	78°	15°	75°
sin						
cos						

2. The results of Exercise 1 suggest that the sine ratio for an acute angle is the same as the cosine ratio for an angle complementary to the given angle. Use the definitions of the sin and cos ratios to show that this is the case. That is, prov that, for each number α such that $0 < \alpha < 90$,

$$\sin \alpha° = \cos (90 - \alpha)°$$

[Notice that words 'sine' and 'cosine' reflect the fact you derived in this exercise.]

3. Pick an acute angle and divide its sin ratio by its cos ratio. U the definition of the sin, cos, and tan ratios to show how you could have predicted this result without actually computing.

4. Pick an acute angle, compute the squares of its sine and cosin ratios, and add the squares. Use the definitions of the sine an cosine ratios to show how you could have predicted this result.

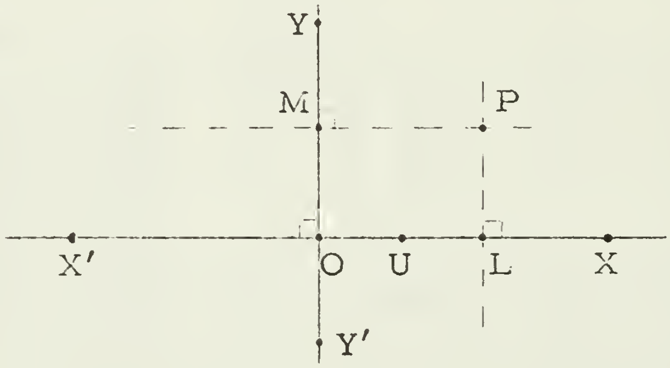
5. Use your knowledge of 30-60-90 triangles and 45-45-90 triangl to derive the sine, cosine, and tangent ratios for angles of 30° 45°, and 60°.

TABLE OF TRIGONOMETRIC RATIOS

Angle	sin	cos	tan	Angle	sin	cos	tan
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875	50°	.7660	.6428	1.1918
6°	.1045	.9945	.1051	51°	.7771	.6293	1.2349
7°	.1219	.9925	.1228	52°	.7880	.6157	1.2799
8°	.1392	.9903	.1405	53°	.7986	.6018	1.3270
9°	.1564	.9877	.1584	54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679	60°	.8660	.5000	1.7321
16°	.2756	.9613	.2867	61°	.8746	.4848	1.8040
17°	.2924	.9563	.3057	62°	.8829	.4695	1.8807
18°	.3090	.9511	.3249	63°	.8910	.4540	1.9626
19°	.3256	.9455	.3443	64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663	70°	.9397	.3420	2.7475
26°	.4384	.8988	.4877	71°	.9455	.3256	2.9042
27°	.4540	.8910	.5095	72°	.9511	.3090	3.0777
28°	.4695	.8829	.5317	73°	.9563	.2924	3.2709
29°	.4848	.8746	.5543	74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002	80°	.9848	.1736	5.6713
36°	.5878	.8090	.7265	81°	.9877	.1564	6.3138
37°	.6018	.7986	.7536	82°	.9903	.1392	7.1154
38°	.6157	.7880	.7813	83°	.9925	.1219	8.1443
39°	.6293	.7771	.8098	84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000				

6.09 Rectangular coordinate systems. --The work you did with ordered pairs of real numbers ["the number plane"] in earlier units can be put to use in doing geometry. You will learn how in this section.

Consider two perpendicular lines $\overleftrightarrow{X'X}$ and $\overleftrightarrow{Y'Y}$ which intersect at the point O between X′ and X and between Y′ and Y. Choose a point U on $\overrightarrow{OX}$.



Given a point P, let L be the foot of the perpendicular through P to $\overleftrightarrow{XX'}$ and let M be the foot of the perpendicular through P to $\overleftrightarrow{YY'}$.

Now, let us assign to P an ordered pair of real numbers. The first of these numbers is called the x-coordinate of P, and the second is called the y-coordinate of P--for short: x(P), and: y(P) [read as 'x of P' and as 'y of P'].

The assignment of coordinates to P is defined as follows:

$$|x(P)| = \frac{OL}{OU}, \text{ and } \begin{cases} x(P) \geq 0 \text{ if } L \in \overrightarrow{OX}, \\ x(P) < 0 \text{ if } L \in \overrightarrow{OX'}. \end{cases}$$

$$|y(P)| = \frac{OM}{OU}, \text{ and } \begin{cases} y(P) \geq 0 \text{ if } M \in \overrightarrow{OY}, \\ y(P) < 0 \text{ if } M \in \overrightarrow{OY'}. \end{cases}$$

This way of assigning ordered pairs of real numbers to points of a plane assures that each point is matched with just one ordered pair and that each ordered pair of real numbers is matched with just one point.

A matching of points with ordered pairs of real numbers obtained in this way is called a rectangular cartesian coordinate system. One obtains such a coordinate system by choosing two perpendicular lines--the x-axis and the y-axis [their point of intersection is called the origin]--and a unit point on the x-axis different from the origin; the matching of ordered pairs of real numbers with points is carried out according to the definition displayed above.

Let's see how this definition works. Referring to the figure, suppose that OU = 8, and that P is a point such that OL = 16 and OM = 24.

(1) What are x(U) and y(U)? (2) What are x(O) and y(O)?

(3) Is x(X) positive or negative? How about y(X)? (4) What are x(P) and y(P)?

Although the definition enables you to answer questions (1), (2), and (3), you don't have enough information to answer question (4). From what you know about P, all you can get from the definition is that

$$|x(P)| = \frac{16}{8} = 2 \qquad \text{and} \qquad |y(P)| = \frac{24}{8} = 3.$$

What more do you need to know about P in order to find x(P) and y(P)?

Let us assume that $L \in \overrightarrow{OX}$ and $M \in \overrightarrow{OY'}$. [Do you see that this assumption tells you that P is in the interior of $\angle XOY'$?] Now, answer question (4).

Recall from your earlier work on measures in section 6.01 that our axioms apply no matter what system, m, of measures you might use. The system used in the present example is one for which $m(\overleftrightarrow{OU}) = 8$, $m(\overleftrightarrow{OL}) = 16$, and $m(\overleftrightarrow{OM}) = 24$. For this system, x(P) = 2 and y(P) = −3. Now, suppose we pick another system of measures, one for which the measure of $\overleftrightarrow{OU}$ is 2. Under that system, what are the measures of $\overleftrightarrow{OL}$ and $\overleftrightarrow{OM}$? Now, answer question (4).

Although, in order to apply the definition to find the coordinates of a point, you must use some system of measures, it evidently doesn't matter what system you use. One thing which does matter, however, is the choice of the unit point. To see this, suppose we select another point as unit point, say, the midpoint, W, of $\overleftrightarrow{OU}$. For the system of measures for which OU = 8, it turns out that OW = 4. With W as unit point,

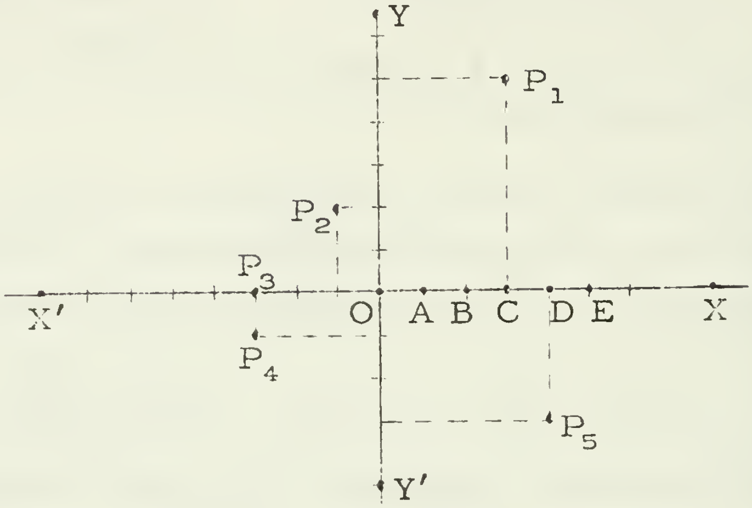
$$|x(P)| = \frac{OL}{OW} \qquad \text{and} \qquad |y(P)| = \frac{OM}{OW}.$$

Since OL = 16 and $L \in \overrightarrow{OX}$, it follows that x(P) = 16/4 = 4. Also, since OM = 24 and $M \in \overrightarrow{OY'}$, it follows that y(P) = −6. [Now, if you use the system of measures for which OU = 2, what are x(P) and y(P)?]

So, the coordinates of a point depend on the choice of unit point. If you halve the length of the <u>unit segment</u>, you double the coordinates of each point. What happens if you double the length of the unit segment?

EXERCISES

A. Here is a picture of a pair of coordinate axes and several points.



1. Use the system of measures for which the measure of $\overleftrightarrow{OA}$ is 1. Then, using A as the unit point, complete the following table:

	P_1	P_2	P_3	P_4	P_5	A	B	C	D	E
x(P)										
y(P)										

2. Using the same system of measures as in Exercise 1, but with C as unit point, complete the following table:

	P_1	P_2	P_3	P_4	P_5	A	B	C	D	E
x(P)										
y(P)										

3. Now, use a system of measures for which $m(\overleftrightarrow{OA}) = 18$. Then, using A as the unit point, complete the following table:

	P_1	P_2	P_3	P_4	P_5	A	B	C	D	E
x(P)										
y(P)										

4. Finally, use a system of measures for which $m(\overleftrightarrow{OA}) = \sqrt{2}$. Then, using A as the unit point, complete the following table:

	P_1	P_2	P_3	P_4	P_5	A	B	C	D	E
x(P)										
y(P)										

B. [Refer to the diagram for Part A.]

1. Consider the system m of measures for which $m(\overleftrightarrow{OA}) = 3$. Use A as unit point and complete the following table:

	P_3	O	A	B	C	D	E
$m(\overleftrightarrow{OP})$							
$\lvert x(P) \rvert$							

2. Now, consider the system d of measures for which $d(\overleftrightarrow{OA}) = 1$, and complete the following table:

	P_3	O	A	B	C	D	E
$d(\overleftrightarrow{OP})$							

*

Do you see that, for each point P on the x-axis, $\lvert x(P) \rvert = d(\overleftrightarrow{OP})$?

For each system of coordinates, and for each point P on the x-axis, $\lvert x(P) \rvert$ is the measure of $\overleftrightarrow{OP}$ in the system of measures for which the measure of the unit segment is 1. Given a coordinate system, we shall denote this system of measures by 'd'.

*

3. Complete each of the following sentences [assuming that A is the unit point and d is the system of measures such that $d(\overleftrightarrow{OA}) = 1$].

(a) $d(\overleftrightarrow{OB}) =$ ______, $\lvert x(B) - x(O) \rvert =$ ______

(b) $d(\overleftrightarrow{AB}) =$ ______, $\lvert x(B) - x(A) \rvert =$ ______

(c) $d(\overleftrightarrow{AD}) =$ ______, $\lvert x(D) - x(A) \rvert =$ ______

(d) $d(\overleftrightarrow{DA}) =$ ______, $\lvert x(A) - x(D) \rvert =$ ______

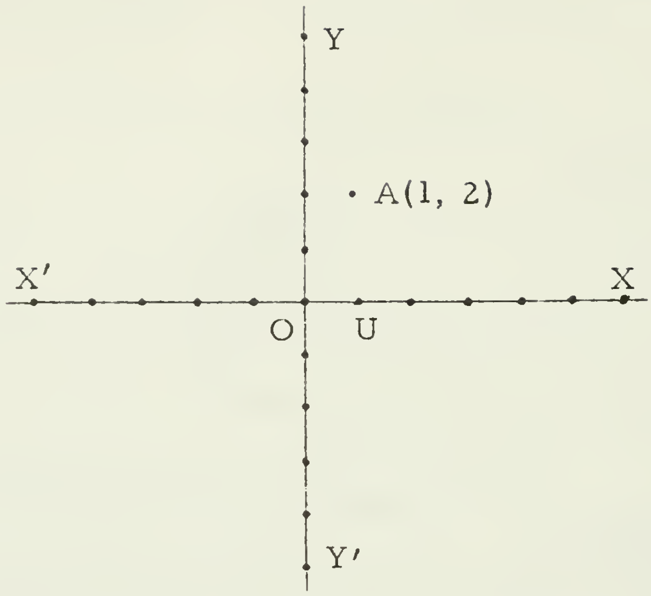
(e) $d(\overleftrightarrow{P_3A}) =$ ______, $\lvert x(A) - x(P_3) \rvert =$ ______

(f) $d(\overleftrightarrow{AP_3}) =$ ______, $\lvert x(P_3) - x(A) \rvert =$ ______

(g) $d(\overleftrightarrow{BE}) =$ ______, $\lvert x(E) - x(B) \rvert =$ ______

(h) $d(\overleftrightarrow{EB}) =$ ______, $\lvert x(B) - x(E) \rvert =$ ______

C.



1. The figure above shows coordinate axes with unit point U. Also, the point A has been plotted and its ordered pair of coordinates indicated. Plot each of the points given below, label the point with its letter name, and indicate its coordinates.

(a) B(2, 2) (b) C(5, 2) (c) D(−1, 2) (d) E(5, −1)

(e) I(−3, 0) (f) G(1, −1) (g) H(−3, −2) (h) F(−1, −1)

2. Complete each of the following sentences. [Remember that d is the system of measures for which $d(\overleftrightarrow{OU}) = 1$.]

(a) $d(\overleftrightarrow{AB}) =$ ____, $|x(B) - x(A)| =$ ____, $|y(B) - y(A)| =$ ____

(b) $d(\overleftrightarrow{BC}) =$ ____, $|x(C) - x(B)| =$ ____, $|y(C) - y(B)| =$ ____

(c) $d(\overleftrightarrow{CA}) =$ ____, $|x(A) - x(C)| =$ ____, $|y(A) - y(C)| =$ ____

(d) $d(\overleftrightarrow{DA}) =$ ____, $|x(A) - x(D)| =$ ____, $|y(A) - y(D)| =$ ____

(e) $d(\overleftrightarrow{BD}) =$ ____, $|x(D) - x(B)| =$ ____, $|y(D) - y(B)| =$ ____

(f) $d(\overleftrightarrow{EG}) =$ ____, $|x(G) - x(E)| =$ ____, $|y(G) - y(E)| =$ ____

(g) $d(\overleftrightarrow{FD}) =$ ____, $|x(D) - x(F)| =$ ____, $|y(D) - y(F)| =$ ____

(h) $d(\overleftrightarrow{EF}) =$ ____, $|x(F) - x(E)| =$ ____, $|y(F) - y(E)| =$ ____

(i) $d(\overleftrightarrow{IU}) =$ ____, $|x(U) - x(I)| =$ ____, $|y(U) - y(I)| =$ ____

(j) $d(\overleftrightarrow{UA}) =$ ____, $|x(A) - x(U)| =$ ____, $|y(A) - y(U)| =$ ____

(k) $d(\overleftrightarrow{AG})$ = ____, $|x(G) - x(A)|$ = ____, $|y(G) - y(A)|$ = ____

(ℓ) $d(\overleftrightarrow{IH})$ = ____, $|x(H) - x(I)|$ = ____, $|y(H) - y(I)|$ = ____

(m) $d(\overleftrightarrow{DF})$ = ____, $|x(F) - x(D)|$ = ____, $|y(F) - y(D)|$ = ____

(n) $d(\overleftrightarrow{FF})$ = ____, $|x(F) - x(F)|$ = ____, $|y(F) - y(F)|$ = ____

(o) $d(\overleftrightarrow{MN})$ = 5, $|x(N) - x(M)|$ = 5, $|y(N) - y(M)|$ = ____

(p) $d(\overleftrightarrow{RS})$ = ____, $|x(S) - x(R)|$ = 0, $|y(S) - y(R)|$ = 8

3. Complete each of the following sentences. [The first has been completed by using the Pythagorean Theorem.]

(a) $d(\overleftrightarrow{AI}) = \sqrt{20}$, $|x(I) - x(A)|$ = 4, $|y(I) - y(A)|$ = 2

(b) $d(\overleftrightarrow{CG})$ = ____, $|x(G) - x(C)|$ = 4, $|y(G) - y(C)|$ = 3

(c) $d(\overleftrightarrow{FI})$ = ____, $|x(I) - x(F)|$ = ____, $|y(I) - y(F)|$ = ____

(d) $d(\overleftrightarrow{DC})$ = ____, $|x(C) - x(D)|$ = ____, $|y(C) - y(D)|$ = ____

(e) $d(\overleftrightarrow{FB})$ = ____, (f) $d(\overleftrightarrow{EG})$ = ____, (g) $d(\overleftrightarrow{EU})$ = ____

4. Sketch each of the sets listed below on the figure you made for Exercise 1.

(a) $\{Z: x(Z) = 2 \text{ and } y(Z) = 0\}$

(b) $\{Z: x(Z) = 2\}$ (c) $\{Z: x(Z) = -3\}$

(d) $\{Z: y(Z) = 3\}$ (e) $\{Z: y(Z) = 0\}$

5. Sketch a pair of coordinate axes and mark a unit point. Then plot the point $P(3, 4)$.

(a) Sketch the set $\{Z: x(Z) = x(P)\}$.

(b) Sketch the set $\{Z: y(Z) = y(P)\}$.

(c) Complete each of the following sentences:

(1) For each real number r, $\{Z: x(Z) = r\}$ is ____ to $\overleftrightarrow{X'X}$.

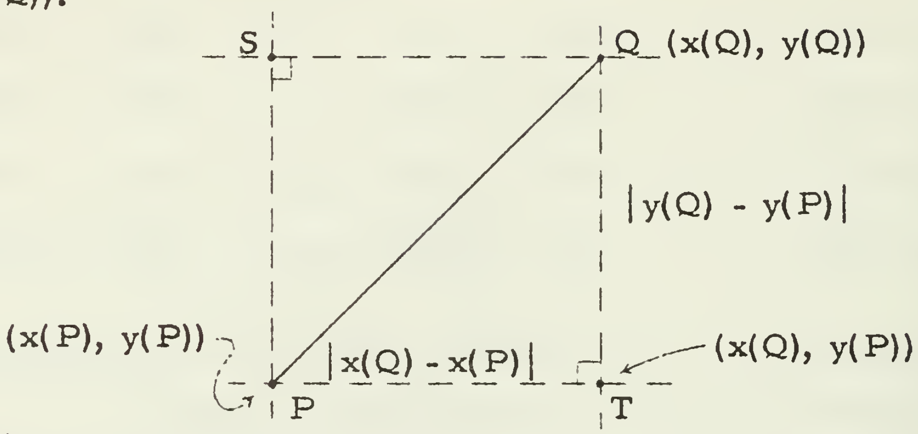
(2) For each real number $r \neq 0$, $\{Z: x(Z) = r\}$ is ____ to $\overleftrightarrow{Y'Y}$.

(3) The perpendicular to $\overleftrightarrow{X'X}$ through P is $\{Z:$ ________ $\}$.

(4) If $P \notin \overleftrightarrow{Y'Y}$, the parallel to $\overleftrightarrow{Y'Y}$ through P is ________.

THE DISTANCE FORMULA

Suppose P and Q are two points with coordinates $(x(P),\ y(P))$ and $(x(Q),\ y(Q))$.



If $\overleftrightarrow{PQ}$ is not perpendicular to the x-axis nor to the y-axis, then the perpendiculars to the axes through P and through Q intersect in the points P, S, Q, and T. These points are the vertices of a rectangle [Why?], and $\overline{PQ}$ is a diagonal of this rectangle.

So, by the Pythagorean Theorem,

$$[d(\overline{PQ})]^2 = [d(\overline{PT})]^2 + [d(\overline{TQ})]^2,$$

or (*) $$d(\overline{PQ}) = \sqrt{[d(\overline{PT})]^2 + [d(\overline{TQ})]^2}$$

Now, since $\overleftrightarrow{PT}$ is the perpendicular to the y-axis through P, and $\overleftrightarrow{QT}$ is the perpendicular to the x-axis through Q, it follows that

$$d(\overline{PT}) = |x(Q) - x(P)| \quad \text{and} \quad d(\overline{TQ}) = |y(Q) - y(P)|.$$

Substituting in (*), we obtain:

$$d(\overline{PQ}) = \sqrt{|x(Q) - x(P)|^2 + |y(P) - y(Q)|^2}$$

The equation enables us to compute the d-measure of a segment given the coordinates of the end points. Does the equation still work if the segment is perpendicular to one of the axes? Why? Does the equation still work if the segment is degenerate?

So, we have the following theorem which is sometimes called <u>the distance formula</u>. [The d-measure of a segment is the distance between its end points when the distance between the origin and the unit point is 1.]

<u>Theorem 9-1.</u> [The Distance Formula]

For each point P, for each point Q,

$$d(\overleftrightarrow{PQ}) = \sqrt{|x(Q) - x(P)|^2 + |y(Q) - y(P)|^2}\ .$$

In using the distance formula we shall often find it convenient to make use of the fact that the square of a real number is nonnegative, and that nonnegative real numbers behave like the numbers of arithmetic. So, we can omit the absolute value bars and use the equation:

$$d(\overleftrightarrow{PQ}) = \sqrt{[x(Q) - x(P)]^2 + [y(Q) - y(P)]^2}$$

EXERCISES

<u>A.</u> Complete the following table:

A	(2, 3)	(1, 1)	(0, 0)	(2, 3)	(2, 2)	(3, −8)	(7, 9)	(m, n)
B	(4, 5)	(4, 5)	(−4, −3)	(7, 15)	(5, 5)	(−3, 0)	(2, −3)	(2, 3)
$d(\overleftrightarrow{AB})$	$2\sqrt{2}$							

<u>B.</u> Set up a coordinate system and plot the points A(2, 3), B(5, 7), and C(9, 4).

1. Prove that $\triangle ABC$ is isosceles, and tell which of its angles is the vertex angle.

2. Compute the perimeter of $\triangle ABC$.

Use cross-section paper for these exercises.

<u>C.</u> 1. Plot the points A(−6, −3), B(3, 9), and C(−3, 1).

2. Show that $B \notin \overleftrightarrow{AC}$. [<u>Hint.</u> Use Axiom A.]

3. Show that $B \in \overleftrightarrow{AC}$.

4. Is $B \in \{Z: d(\overleftrightarrow{ZC}) > d(\overleftrightarrow{AC})\}$?

D. 1. Plot the points A(5, 5), B(7, 1), and C(−3, 1).

2. Show that ΔABC is a right triangle.

3. Do Exercise 2 another way.

4. Prove that DEFG [D(15, 8), E(16, 3), F(−9, −2), G(−10, 3)] is a rectangle.

5. Find the coordinates of the point of intersection of the diagonals of DEFG.

E. 1. Prove that the four points with coordinates (−2, 5), (−5, 1), (−1, −2), and (2, 2) are the vertices of a square.

2. Prove that the point with coordinates (2, 7) belongs to the perpendicular bisector of $\overleftrightarrow{AB}$ [A(3, 4), B(−1, 7)].

3. If ABCD is a parallelogram and the coordinates of A, B, and C are (1, 2), (3, 5), and (8, 5), respectively, what are the coordinates of D?

4. Guess the coordinates of the point of intersection of the diagonals of the parallelogram in Exercise 3.

* * *

Example. Compute the distance between $M\left(\frac{a-1}{2}, \frac{a}{2}\right)$ and $N\left(\frac{a+1}{2}, -\frac{a}{2}\right)$.

Solution.
$$d(\overleftrightarrow{MN}) = \sqrt{\left(\frac{a+1}{2} - \frac{a-1}{2}\right)^2 + \left(-\frac{a}{2} - \frac{a}{2}\right)^2}$$
$$= \sqrt{\left(\frac{a+1-(a-1)}{2}\right)^2 + \left(\frac{-a-a}{2}\right)^2}$$
$$= \sqrt{\left(\frac{2}{2}\right)^2 + \left(\frac{-2a}{2}\right)^2}$$
$$= \sqrt{1 + a^2}$$

* * *

F. Compute the distance between each two points.

1. $A\left(\frac{m}{2}, \frac{n}{3}\right)$ and $B(m, 2n)$ 2. $C\left(\frac{a-b}{2}, \frac{a+b}{2}\right)$ and $D\left(\frac{a}{2}, \frac{b}{2}\right)$

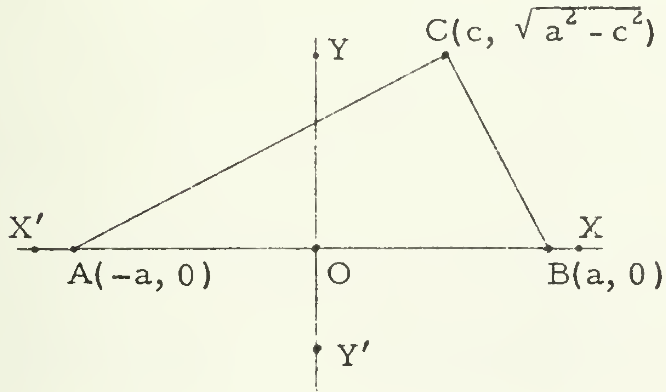
3. $E(2a + 1, 3b - 2)$ and $F(3a - 2, 4b + 5)$

4. $G(x + 3y, 2x - 5y)$ and $H(2x - y, x + 4y)$

5. $I\left(\frac{4-a}{2}, \frac{b}{2}\right)$ and $J(1, b)$

6. $K\left(\frac{a}{2}, \frac{b}{2}\right)$ and $L\left(\frac{4-a}{2}, \frac{b}{2}\right)$

G. 1.



Given: $a > 0$, $0 \leq c \leq a$.

Find: $d(\overleftrightarrow{OC})$, $d(\overleftrightarrow{AC})$, $d(\overleftrightarrow{BC})$, $d(\overleftrightarrow{AB})$, $m(\angle ACB)$

2. Repeat Exercise 1 for a point $D\left(d, \sqrt{a^2 - d^2}\right)$, $0 \leq d \leq a$.

3. Repeat for a point $E\left(\frac{a}{2}, \frac{a}{2}\sqrt{3}\right)$. Then, find $m(\angle EAB)$.

EXPLORATION EXERCISES

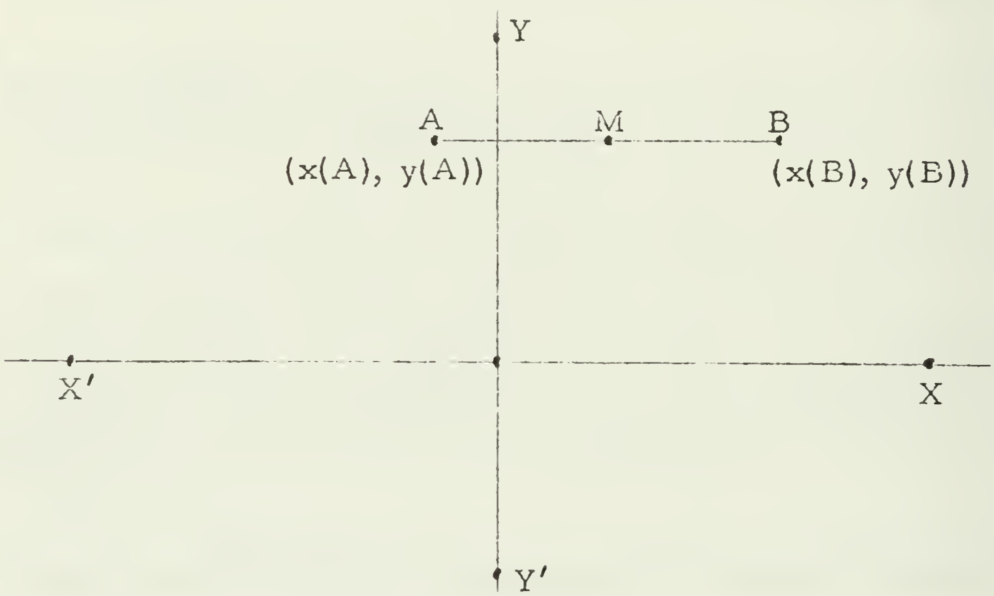
Plot each pair of points and compute the coordinates of the midpoint of the segment joining the points.

1. A(1, 0) and B(6, 0)
2. C(-3, 0) and D(7, 0)
3. E(0, 5) and F(0, 11)
4. G(0, -2) and H(0, -5)
5. I(2, 5) and J(2, 8)
6. K(3, 7) and L(-4, 7)
7. M(3, 3) and N(7, 7)
8. P(-5, 5) and Q(3, -3)
9. R(0, 0) and S(6, 8)
10. T(-4, -3) and V(8, 6)
11. A'(4, 3) and B'(10, 7)
12. C'(5, 7) and D'(1, 4)
13. E'(2, -5) and F'(8, 7)
14. G'(-2, -3) and H'(9, -5)

THE MIDPOINT FORMULA

Consider the segment whose end points are A and B. Let us try to derive a formula which can be used to compute the coordinates of the midpoint, M, from the coordinates of A and B.

We first consider the case of a segment which is perpendicular to the y-axis. Once we have done this, it will be easy to deal with the case of a segment perpendicular to the x-axis. Then, we shall use these results to handle the remaining case of a segment which is not perpendicular to either axis.



Our job is to compute x(M) and y(M).

By the definition of midpoint, M is the point which belongs to $\overline{AB}$ and which is equidistant from A and B.

Since $\overleftrightarrow{AB}$ is perpendicular to the y-axis, all points of $\overleftrightarrow{AB}$ have the same y-coordinate. So, y(A) = y(M) = y(B). Thus, we have found the y-coordinate of M.

Again, since $\overleftrightarrow{AB}$ is perpendicular to the y-axis, M is equidistant from A and B if and only if

$$|x(M) - x(A)| = |x(B) - x(M)|.$$

Finally, since $M \in \overline{AB}$ [because $A \neq B$], it can be shown from Axioms A and B and properties of the absolute value operation that x(M) - x(A) and x(B) - x(M) are either both positive or both negative. So, M is equidistant from A and B if and only if

$$x(M) - x(A) = x(B) - x(M).$$

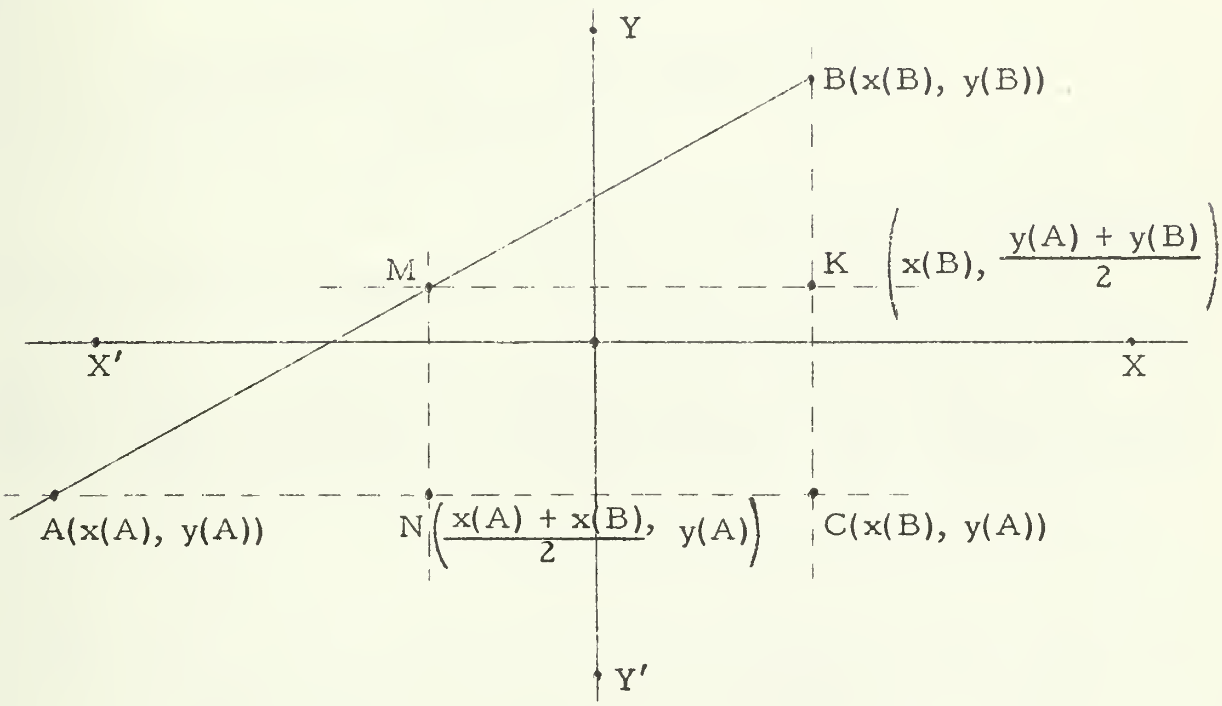
Solving this last equation for '$x(M)$', we get:

$$x(M) = \frac{x(A) + x(B)}{2}$$

This gives us the x-coordinate of M.

Now, what if $\overline{AB}$ is a segment perpendicular to the x-axis? How do you compute the coordinates of its midpoint?

Let us now consider the third case, that of a segment which is perpendicular to neither axis.



Consider the lines through M and B perpendicular to the x-axis and the lines through M and A perpendicular to the y-axis. Since M is the midpoint of $\overline{AB}$ and $\overleftrightarrow{MN} \parallel \overleftrightarrow{BC}$, it follows that N is the midpoint of $\overline{AC}$. [Why?] So, since $\overline{AC}$ is perpendicular to the y-axis, it follows that

$$x(N) = \frac{x(A) + x(C)}{2}.$$

But, since $\overleftrightarrow{MN}$ and $\overleftrightarrow{BC}$ are perpendicular to the x-axis, it follows that

$$x(N) = x(M) \text{ and } x(C) = x(B).$$

So,

$$(1) \qquad x(M) = \frac{x(A) + x(B)}{2}.$$

Similarly,

$$(2) \qquad y(M) = \frac{y(A) + y(B)}{2}.$$

Notice that formulas (1) and (2) both hold for segments perpendicular to either of the coordinate axes. [Do they hold if $A = B$?]

So, we have the following theorem:

<u>Theorem 9-2.</u> [The Midpoint Formula]

$\forall_P \forall_Q \forall_M$

M is the midpoint of $\overline{PQ}$

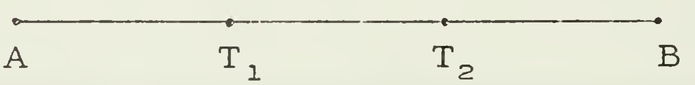
if and only if

$$x(M) = \frac{x(P) + x(Q)}{2} \text{ and } y(M) = \frac{y(P) + y(Q)}{2}$$

POINTS OF DIVISION OF SEGMENTS

The midpoint of a segment $\overline{AB}$ is the point which divides the segment from A to B in the ratio 1 to 1.

Now, consider the segment $\overline{AB}$ and the points T_1 and T_2 of $\overline{AB}$ where $\overline{AT} \cong \overline{T_1T_2} \cong \overline{T_2B}$. These points are <u>the trisection points</u> of $\overline{AB}$. Notice



that T_1 divides the segment from A to B in the ratio 1 to 2, and T_2 divides the segment from A to B in the ratio 2 to 1.

EXERCISES

<u>A.</u> Each of the following exercises lists the end points of a segment. Use the midpoint formula to compute the coordinates of the midpoints of the segments.

1. A(3, 2) and B(7, 6)
2. C(5, 1) and D(8, −3)
3. E(8, 5) and F(12, 5)
4. G(0, −8) and H(8, 0)
5. I(a, b) and J(a + c, b + c)
6. K(a + c, b + d) and L(a + e, b)

B. Find the trisection points of $\overleftrightarrow{AB}$ if the coordinates of A are (2, 3) and those of B are (8, 6).

C. Derive formulas for computing the coordinates of the trisection points of $\overleftrightarrow{AB}$ from the coordinates of A and B.

D. 1. Show that the point which divides the segment from A to B in the ratio p to q has the coordinates

$$\left(\frac{q \cdot x(A) + p \cdot x(B)}{p + q}, \ \frac{q \cdot y(A) + p \cdot y(B)}{p + q}\right).$$

2. Show that these coordinates are

$$\left(x(A) + \frac{p}{p + q}[x(B) - x(A)], \ y(A) + \frac{p}{p + q}[y(B) - y(A)]\right).$$

3. Check these formulas in the case of the midpoint of a segment, and in the case of the trisection points of a segment.

4. (a) Use the formula in Exercise 2 to find the point which is 4/7 of the way from A(2, 5) to B(9, 17). Before using the formula be sure you have figured out appropriate values of 'p' and 'q'.

(b) Now, derive a formula for the coordinates of the point $P \in \overleftrightarrow{AB}$ such that $AP/AB = r$. [P is the point which is "r of the way" from A to B.]

E. 1. Find the measures of the medians of the triangle whose vertices are A(−2, −5), B(−6, 7), and C(2, 3).

2. Show that the points A(6, 1), B(2, 3), C(−2, −4), and D(2, −6) are vertices of a parallelogram. [Hint. Use the theorem about diagonals of a quadrilateral bisecting each other.]

3. Are the points A(900, 843), B(906, 831), C(918, 807), and D(921, 801) the vertices of a parallelogram?

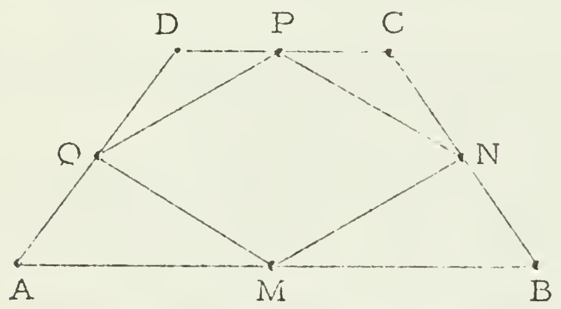
4. The points A(0, 0), B(4, 1), and C(3, 5) are vertices of the parallelogram ABCD. Find the coordinates of D.

5. The points A(0, 0), B(4, 1), and C(3, 5) are vertices of a parallelogram. Find the coordinates of the fourth vertex.

PROOFS BY ANALYTIC GEOMETRY

Some of the theorems you have already proved in earlier sections as well as theorems you have not yet proved can be handled by using coordinate systems. Proofs which use this approach are often called <u>proofs by analytic geometry</u> or <u>analytic proofs</u>. The sort of proofs you have given earlier are sometimes called <u>synthetic proofs</u>. Of course, all proofs depend on the same rules of reasoning, and both synthetic and analytic proofs deal with the same subject matter, geometry. The only difference between the two kinds of proof is that analytic proofs use coordinate systems and synthetic proofs do not.

<u>Example 1.</u>



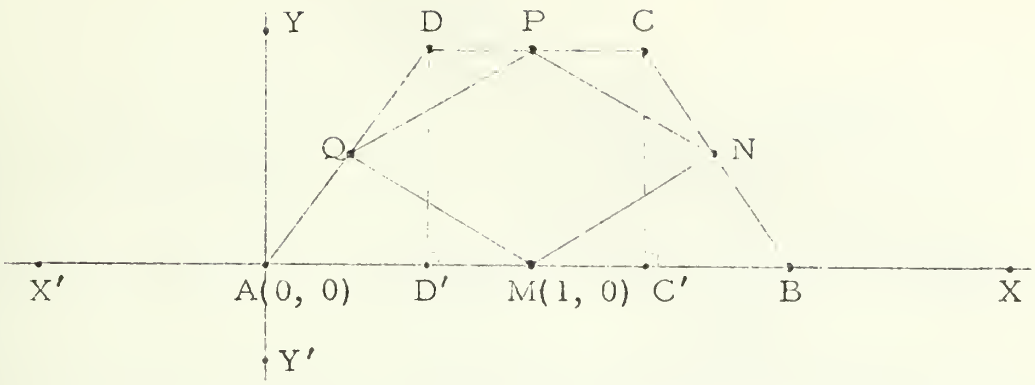
<u>Hypothesis:</u> ABCD is an isosceles trapezoid with AD = CB, M, N, P, and Q are midpoints of the sides

<u>Conclusion:</u> MNPQ is a rhombus

<u>Plan.</u> Choose a coordinate system and use the distance formula to show that PQ = QM = MN = NP. Then, use the definition of a rhombus.

<u>Discussion.</u> As you have seen, you can set up a system of rectangular coordinates by choosing any line for the x-axis, and any two points on this line for the origin and unit point. The y-axis is then determined--it is the perpendicular to the x-axis through the origin. [It is customary to choose the positive direction on the y-axis in such a way that if you "walk" from (0, 0) to (1, 0) to (0, 1) to (0, 0) you will have walked in the counterclockwise direction.]

It is helpful to choose a coordinate system which "fits" the geometric figure you are discussing--that is, which takes advantage of some of the properties of the figure. For example, in this case, it will be helpful to choose $\overleftrightarrow{AB}$ as x-axis, and to choose some two of the three points A, M, and B, as origin and unit points. Let's do this, choosing A as origin and M as unit point.



Now, we assign coordinates to various points of the figure in accordance with the hypothesis.

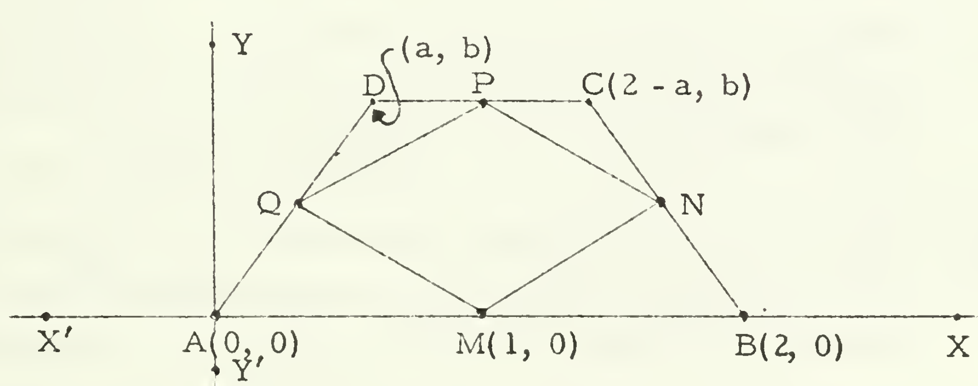
Since M is the midpoint of $\overleftrightarrow{AB}$, B has coordinates (2, 0).

Since $\overleftrightarrow{DC}$ is parallel to the x-axis, D and C have the same y-coordinate. Suppose the common y-coordinate of D and C is b.

Now, let's assign x-coordinates to D and C. Consider the perpendiculars from D and C to the x-axis. If D′ and C′ are the feet of these perpendiculars then, since $\overleftrightarrow{AD} \cong \overleftrightarrow{BC}$ and $\overleftrightarrow{D'D} \cong \overleftrightarrow{C'C}$, it follows by the hypotenuse-leg theorem that $ADD' \longleftrightarrow BCC'$ is a congruence. So, $\overleftrightarrow{D'A} \cong \overleftrightarrow{C'B}$.

Suppose $d(\overleftrightarrow{D'A}) = a$. Then, the x-coordinate of D is a. And since $d(\overleftrightarrow{C'B}) = a$ and $d(\overleftrightarrow{AB}) = 2$, it follows that $d(\overleftrightarrow{C'A}) = 2 - a$. So, the x-coordinate of C is 2 - a.

<u>Solution.</u>

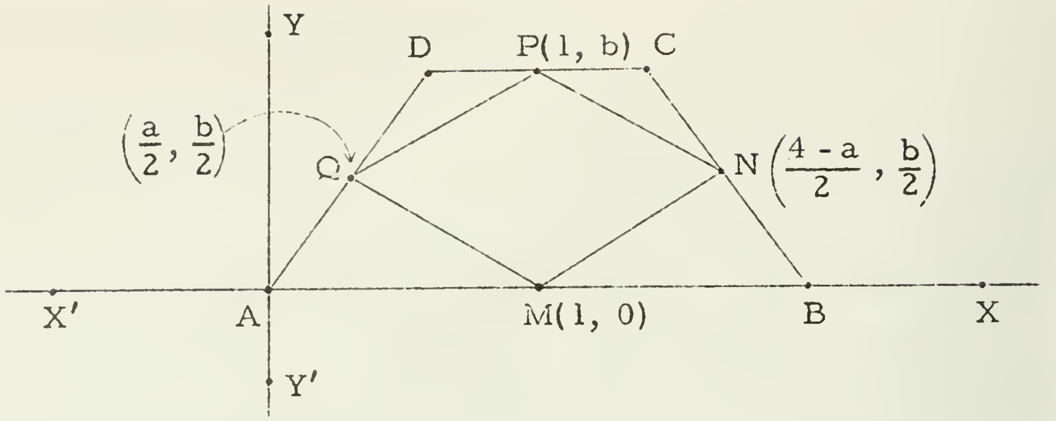


By the midpoint formula,

the coordinates of N are $\left(\frac{(2-a)+2}{2}, \frac{b+0}{2}\right)$, or $\left(\frac{4-a}{2}, \frac{b}{2}\right)$,

the coordinates of P are $\left(\frac{a+(2-a)}{2}, \frac{b+b}{2}\right)$, or (1, b),

and the coordinates of Q are $\left(\frac{0+a}{2}, \frac{0+b}{2}\right)$, or $\left(\frac{a}{2}, \frac{b}{2}\right)$.



By the distance formula,

$$d(\overline{PQ}) = \sqrt{\left(\frac{a}{2} - 1\right)^2 + \left(\frac{b}{2} - b\right)^2} = \sqrt{\left(\frac{a-2}{2}\right)^2 + \left(-\frac{b}{2}\right)^2}$$

$$d(\overline{QM}) = \sqrt{\left(1 - \frac{a}{2}\right)^2 + \left(0 - \frac{b}{2}\right)^2} = \sqrt{\left(\frac{2-a}{2}\right)^2 + \left(-\frac{b}{2}\right)^2}$$

$$d(\overline{MN}) = \sqrt{\left(\frac{4-a}{2} - 1\right)^2 + \left(\frac{b}{2} - 0\right)^2} = \sqrt{\left(\frac{2-a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$$

$$d(\overline{NP}) = \sqrt{\left(1 - \frac{4-a}{2}\right)^2 + \left(b - \frac{b}{2}\right)^2} = \sqrt{\left(\frac{a-2}{2}\right)^2 + \left(\frac{b}{2}\right)^2}.$$

Since the square of a number is the square of its opposite, the four sides of MNPQ are congruent. So, MNPQ is a rhombus.

Note 1. Another plan for proving that MNPQ is a rhombus is to show that $\overline{MP}$ and $\overline{NQ}$ have the same midpoint and are perpendicular Then use Theorem 6-17. Carry out this plan.

Note 2. Sometimes you can assign coordinates in such a way as to avoid fractions. In this case, suppose that $d(\overline{D'A}) = 2a$ and that the y-coordinate of D is 2b. Carry out the details.

Note 3. Another way of getting a simple analytic proof is to make use of whatever "symmetry" the geometric figure has. We could do so in this case by choosing M as origin and B as unit point. Now, if we suppose that the coordinates of C are (a, b) then, since $\overline{D'A} \cong \overline{C'B}$ and $\overline{AM} \cong \overline{MB}$, the coordinates of D are $(-a, b)$.

Example 2. Draw three pictures of rhombus ABCD. Set up three coordinate systems.

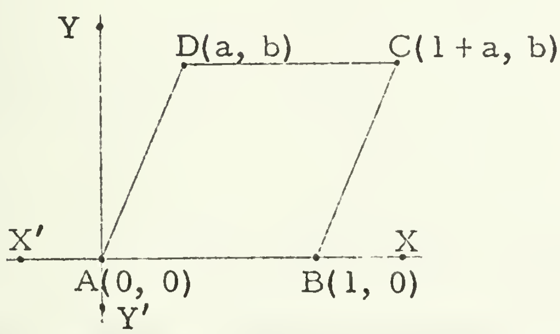
(I) $\overleftrightarrow{AB}$ = x-axis, A = origin, and B = unit point

(II) $\overleftrightarrow{AC}$ = x-axis, A = origin, and C = unit point

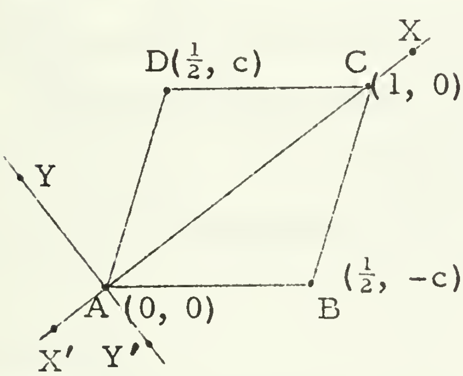
(III) $\overleftrightarrow{AC}$ = x-axis, midpoint of $\overline{AC}$ = origin, and C = unit point

In each case assign coordinates to all four vertices.

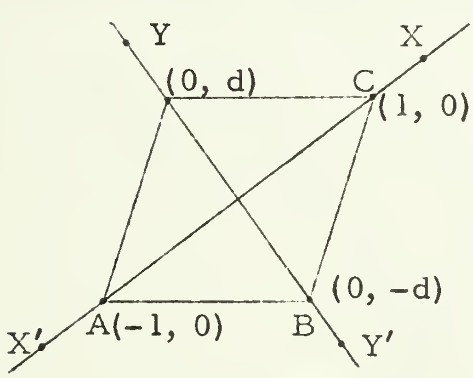
Solution.



Case (I) $\overleftrightarrow{AB}$ = x-axis,
A = origin,
B = unit point



Case (II) $\overleftrightarrow{AC}$ = x-axis,
A = origin,
C = unit point



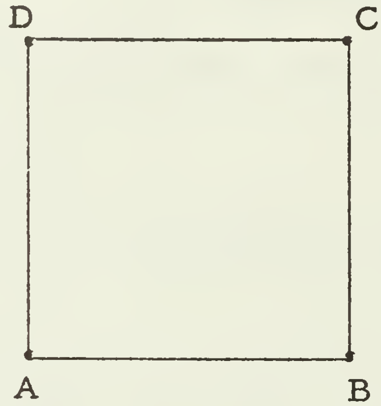
Case (III) $\overleftrightarrow{AC}$ = x-axis,
midpoint of $\overline{AC}$ = orig
C = unit point

Note. You may have noticed that the assignment of coordinates in Case (I) would work for any parallelogram as well as for a rhombus. What additional condition is needed to express the hypothesis that ABCD is a rhombus? [Hint. What is $d(\overline{AD})$?]

EXERCISES

A. Each exercise shows a geometric figure and describes some coordinate systems. For each description, draw a copy of the figure, set up the coordinate system, and assign coordinates to the vertices.

1. A square



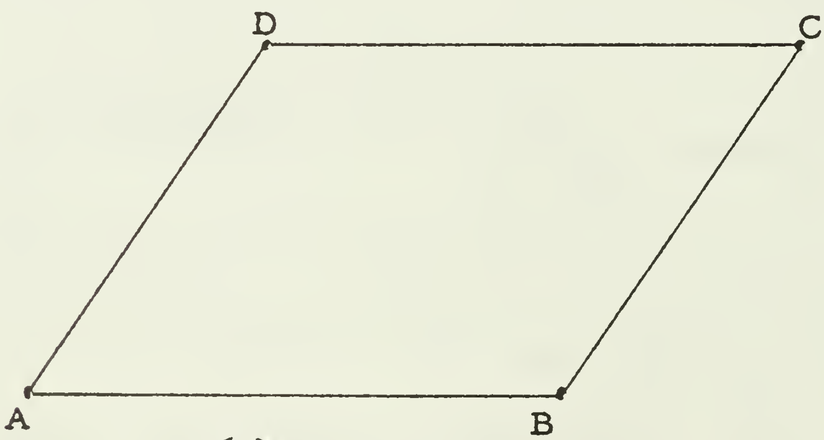
(I) x-axis = $\overleftrightarrow{AB}$, origin = A, unit point = B

(II) x-axis = $\overleftrightarrow{AB}$, origin = midpoint of $\overline{AB}$, unit point = B

(III) x-axis = line joining the midpoints of $\overline{AB}$ and $\overline{CD}$,
y-axis = line joining the midpoints $\overline{AD}$ and $\overline{BC}$,
unit point = midpoint of $\overline{CD}$

(IV) x-axis = $\overleftrightarrow{AC}$, y-axis = $\overleftrightarrow{BD}$, unit point = C

2. A parallelogram

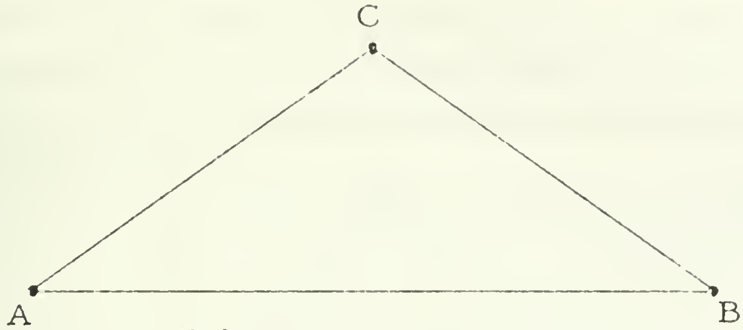


(I) x-axis = $\overleftrightarrow{AB}$, origin = A, unit point = B

(II) x-axis = $\overleftrightarrow{AD}$, origin = D, unit point = A

(III) x-axis = $\overleftrightarrow{AC}$, the midpoint of $\overline{AC}$ belongs to the y-axis, unit point = C

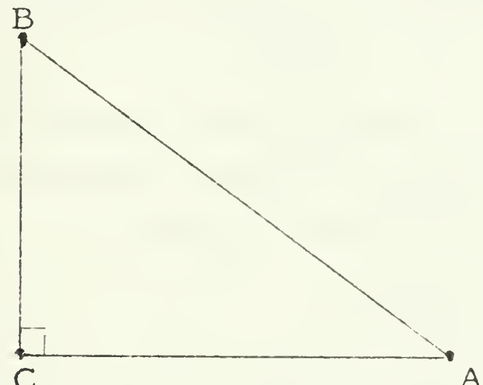
3. An isosceles triangle



(I) x-axis = $\overleftrightarrow{AB}$, origin = A, unit point = B

(II) x-axis = $\overleftrightarrow{AB}$, origin = midpoint of $\overline{AB}$, unit point = B

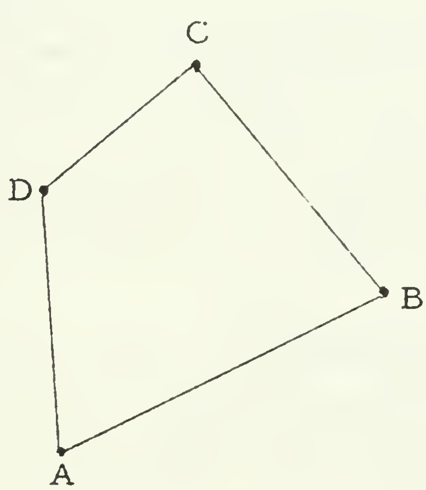
4. A right triangle



(I) x-axis = $\overleftrightarrow{AC}$, y-axis = $\overleftrightarrow{BC}$, unit point = A

(II) x-axis = $\overleftrightarrow{BC}$, origin = B, unit point = C

5. A quadrilateral

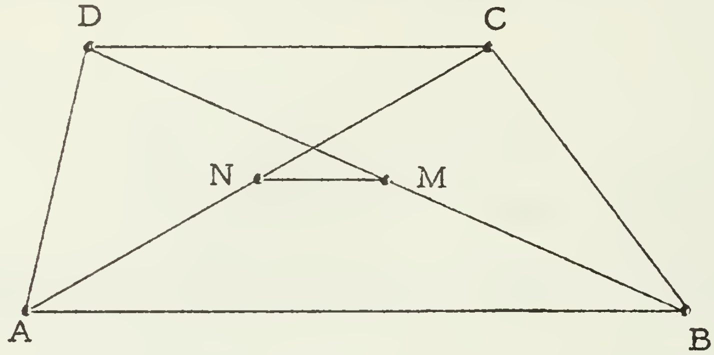


(I) x-axis = $\overleftrightarrow{AB}$, origin = A, unit point = B

(II) x-axis $\cap$ ABCD = $\emptyset$, y-axis $\cap$ ABCD = $\emptyset$

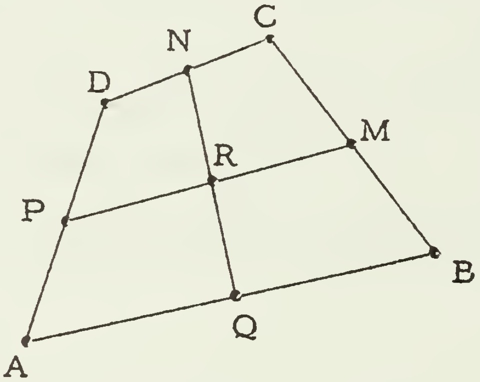
B. Use analytic geometry to prove the following theorems.

1. The segment joining the midpoints of the diagonals of a trapezoid is parallel to the bases of the trapezoid, and its measure is half the difference of the measures of the bases.

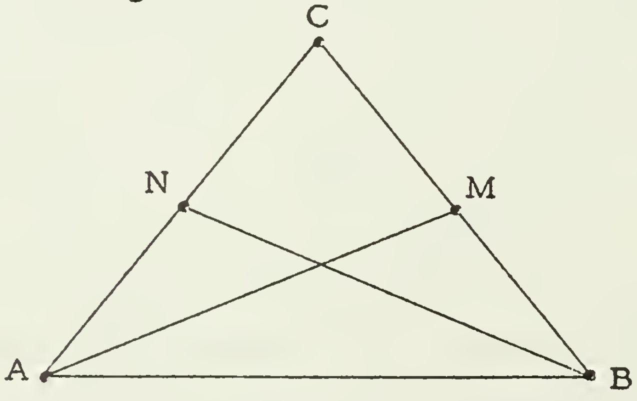


2. The midpoint of the hypotenuse of a right triangle is equidistant from all three vertices of the triangle.

3. The segments joining the midpoints of the opposite sides of a quadrilateral bisect each other. [Use the coordinate system described in Exercise 5(II) of Part A, above.]

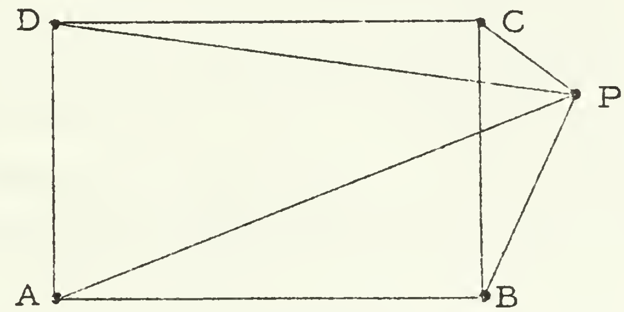


4. If ΔABC is isosceles with vertex angle $\angle C$, then the medians to the legs are congruent.

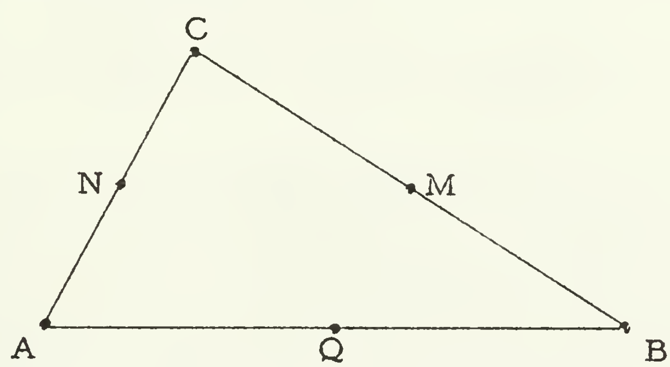


5. If two medians of a triangle are congruent, the triangle is isosceles.

6. The sum of the squares of the distances between a point and each of two opposite vertices of a rectangle is the sum of the squares of the distances between the point and each of the other two vertices. [Does the point have to belong to the "exterior" of ABCD?]



7. Four times the sum of the squares of the measures of the medians of a triangle is three times the sum of the squares of the measures of the sides.



8. If the diagonals of a trapezoid are congruent then the trapezoid is isosceles.

<u>C.</u> 1. Suppose that ΔABC is a right triangle with right angle $\angle C$, and that the coordinates of A and B are $(0, 0)$ and $(20, 0)$ respectively. If the x-coordinate of C is 16, what is its y-coordinate?

2. Suppose that ΔABC is right-angled at $C(0, b)$, that B is the unit point, and that A is on the x-axis. What are the coordinates of A?

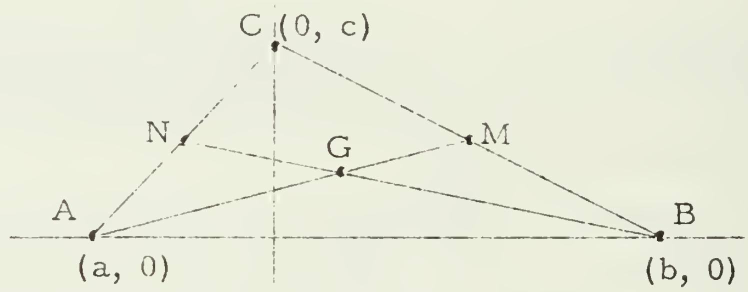
3. Suppose that ΔABC is an isosceles triangle with vertex angle $\angle C$, and that the unit point is C. If the coordinates of B are $(0, b)$ and A is on the x-axis, what are the coordinates of A?

4. Suppose that ABCDEF is a regular hexagon. If A is the origin, B the unit point, and C has a positive y-coordinate, what are the coordinates of C, D, E, and F?

MEDIANS OF A TRIANGLE

In an earlier section you proved that the angle bisectors of a triangle are concurrent, that is, that all three angle bisectors contain the same point. You also proved that this is the case for the perpendicular bisectors of the sides of a triangle, and also for the altitudes. Is it the case for the medians? The answer to this question is 'yes' and the proof by analytic geometry shows the power of this technique.

Consider a triangle $\triangle ABC$, and set up a coordinate system as shown in the figure. Now, the medians $\overleftrightarrow{AM}$ and $\overleftrightarrow{BN}$ intersect at some



point G. Let's see what we can find out about this point.

Suppose that $AG/AM = r$ and $BG/BN = s$. The numbers r and s are numbers between 0 and 1 [Explain]. We are now going to show that $r = s$.

In Exercise 4(b) of Part D on page 6-245, you were asked to derive a formula for the coordinates of such a point as G. Using that formula we find that the coordinates of G are

$$(1)\quad \Big(x(A) + r[x(M) - x(A)],\ y(A) + r[y(M) - y(A)]\Big).$$

But, the coordinates of G are also

$$(2)\quad \Big(x(B) + s[x(N) - x(B)],\ y(B) + s[y(N) - y(B)]\Big).$$

Since M is the midpoint of $\overleftrightarrow{BC}$,

$$x(M) = \frac{b}{2} \quad\text{and}\quad y(M) = \frac{c}{2}.$$

Similarly,

$$x(N) = \frac{a}{2} \quad\text{and}\quad y(N) = \frac{c}{2}.$$

Substituting in (1) and (2), we obtain:

$$(1')\quad \left(a + r(\frac{b}{2} - a),\ 0 + r(\frac{c}{2} - 0)\right)$$

and:

$$(2')\quad \left(b + s(\frac{a}{2} - b),\ 0 + s(\frac{c}{2} - 0)\right)$$

Since (1′) and (2′) are just different expressions for the coordinates of the same point, it follows that

$$(3) \qquad a + r\left(\frac{b}{2} - a\right) = b + s\left(\frac{a}{2} - b\right)$$

and

$$(4) \qquad 0 + r\left(\frac{c}{2} - 0\right) = 0 + s\left(\frac{c}{2} - 0\right).$$

Now, since A, B, and C are vertices of a triangle, $c \neq 0$. So, equation (4) tells us that

$$(5) \qquad r = s.$$

We can get more information by substitution from (5) into (3):

$$a + s\left(\frac{b}{2} - a\right) = b + s\left(\frac{a}{2} - b\right)$$

Solving for 's':

$$s\left(\frac{b}{2} - a\right) - s\left(\frac{a}{2} - b\right) = b - a$$

$$s\left[\left(\frac{b}{2} - a\right) - \left(\frac{a}{2} - b\right)\right] = b - a$$

$$s\left[\frac{3}{2}b - \frac{3}{2}a\right] = b - a$$

$$\frac{3}{2}s(b - a) = b - a$$

So, since $a \neq b$ [Why?],

$$(6) \qquad s = \frac{2}{3}.$$

Equations (5) and (6) show that the point of intersection of any two medians of a triangle divides each of them in the ratio 2 to 1. So, for example, the median of ΔABC from A intersects the median from C at the point which divides the latter in the ratio 2 to 1. And, the median from B intersects the median from C at this same point. Consequently, the medians of ΔABC are concurrent.

Theorem 9-3.
The medians of a triangle are concurrent at a point which divides each median from the vertex to the midpoint of the opposite side in the ratio 2 to 1.

EXERCISES

A. Use the technique illustrated above to show that the diagonals of a parallelogram bisect each other.

B. Consider a parallelogram ABCD. Let R be the midpoint of $\overline{AD}$ and let T be the midpoint of $\overline{CD}$. Suppose $\overline{AC}$ crosses $\overline{BR}$ at P and $\overline{BT}$ at Q. Guess the ratio of $\overline{AP}$ to $\overline{AC}$, and prove your guess using either analytic geometry or synthetic geometry.

C. 1. Consider a right triangle ΔABC whose legs are 18 units and 24 units long. If $\overline{CM}$ is the median to the hypotenuse and $\overline{BN}$ is the median from B and $\overline{CM} \cap \overline{BN} = \{G\}$, find the length of $\overline{CG}$.

2. Suppose the measure of a side of an equilateral triangle is s. Use s to compute the distance between the point of concurence of the medians and a vertex.

3. Suppose ΔABC is an isosceles right triangle with right angle $\angle C$. If the measure of a leg is s, what are the distances between the point of concurrence of the medians and each vertex?

4. (a) Suppose the vertices of ΔABC have coordinates (a_1, a_2), (b_1, b_2), and (c_1, c_2). Compute the coordinates of the point of concurrence of the medians.

(b) Compute the coordinates of the point of concurrence of the medians of ΔABC for A(2, 1), B(10, 3), and C(0, 11).

D. 1. If $\overleftrightarrow{PQ}$ intersects the x-axis at R, and the coordinates of P and Q are (3, −3) and (10, 4), what are the coordinates of R?

2. If, in Exercise 1, S is a point of $\overleftrightarrow{PQ}$ and the x-coordinate of S is 12, what is the y-coordinate of S?

EXPLORATION EXERCISES

A. 1. Plot the point A(5, 5) and the point B(0, 0). What is the measure of $\angle ABX$?

2. Find the measure of $\angle ABX$ if the coordinates of A are (−5, 5) and those of B are (0, 0). [What is the measure of $\angle ABX'$?]

3. Complete the following table. [Hint. You may find it convenient to use your knowledge of the tangent ratio for an angle, and the table on page 6-231.]

A	$(2, 2\sqrt{3})$	$(-2, 2\sqrt{3})$	$(5\sqrt{3}, 5)$	$(-5\sqrt{3}, 5)$	(2, 7)	(-5, 3)
B	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
m(∠ABX)						
m(∠ABX′)						

4. Find the measure of ∠ABX if the coordinates of A are (5, 2) and those of B are (3, 0).

5. Find the measure of ∠ABX if the coordinates of A are (1, 2) and those of B are (3, 0).

B. 1. Plot the points A(5, 4), C(7, 8), and D(7, 4). What is the measure of ∠CAD?

2. Plot the points A and C of Exercise 1. Let B be the point of intersection of $\overleftrightarrow{AC}$ and the x-axis. What is m(∠ABX)?

3. If the coordinates of A and C are (1, 2) and (6, 8), respectively, and $\overleftrightarrow{AC} \cap$ x-axis = {B}, what is m(∠ABX)?

4. If the coordinates of A and C are (−3, 8) and (2, 3), respectively, and $\overleftrightarrow{AC} \cap$ x-axis = {B}, what is m(∠ABX)?

5. Complete the following table. [B is the point of intersection of $\overleftrightarrow{AC}$ and the x-axis.]

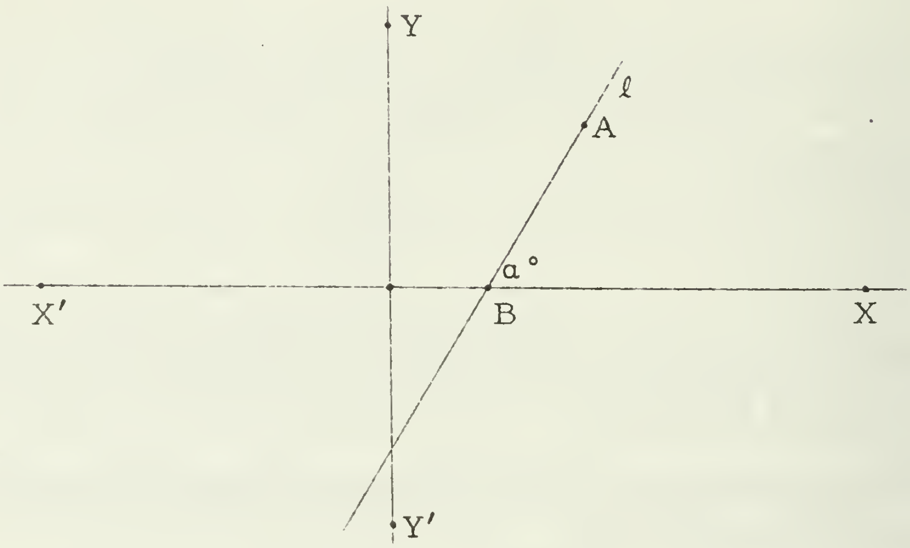
A	$(7, -2 + 5\sqrt{3})$	(7, 8)	(−8, 5)	(−3, 4)	(−4, 5)
C	(2, −2)	(2, −2)	(2, −2)	(4, −3)	(−3, 0)
m(∠ABX)					
m(∠ABX′)					

6. Suppose that A is a point whose second coordinate is positive, and that B is a point on the x-axis. If $\overleftrightarrow{AB}$ contains the points C(5, −2) and D(3, −4), what is the measure of ∠ABX?

7. Repeat Exercise 6 for the points C(−6, −1) and D(−4, −4).

SLOPE OF A LINE

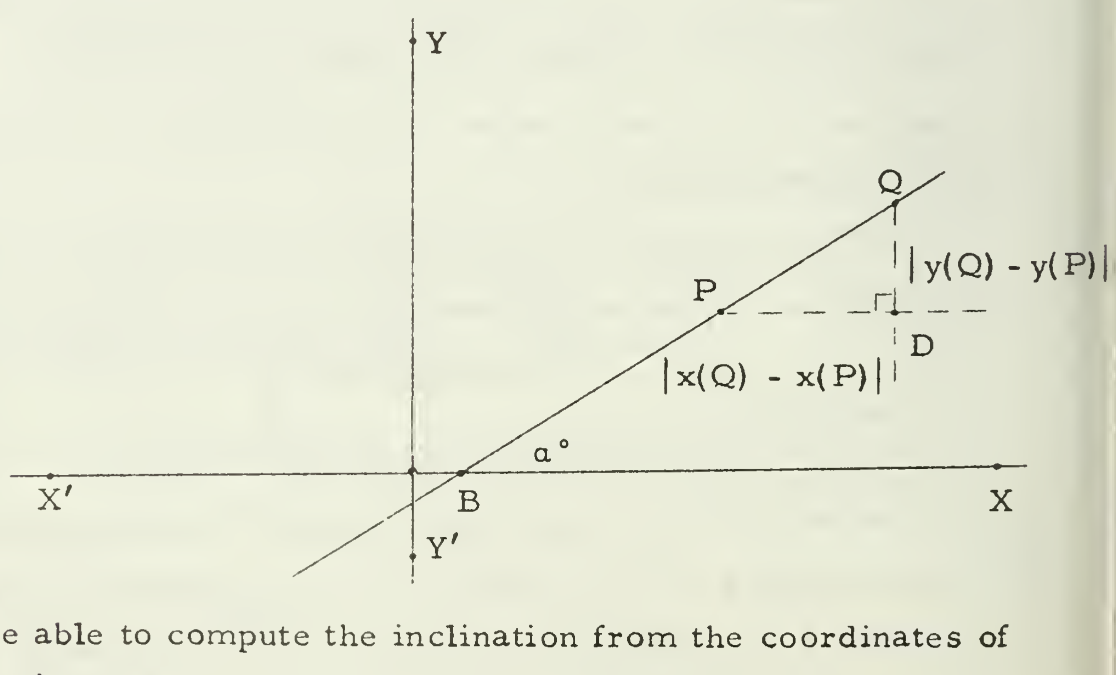
Suppose that we have a coordinate system such that the line ℓ is perpendicular to neither axis--that is, the line ℓ is oblique. Now, if



A is on the Y-side of the x-axis, and B is the point of intersection of ℓ and the x-axis, then $m(\angle ABX)$ is called the inclination of ℓ.

Let $m(\angle ABX) = a$. If P is any point in the Y-side of the x-axis then $P \in \ell$ if and only if $m(\angle PBX) = a$. [Why?] Also, if P is any point in the other side of the x-axis then $P \in \ell$ if and only if $m(\angle PBX') =$ _____.

Now, let us suppose that we have a coordinate system, that P and Q are two points that determine an oblique line. Further, suppose that the inclination of $\overleftrightarrow{PQ}$ is less than 90.



We should be able to compute the inclination from the coordinates of P and Q. Let's do so.

Since $\overrightarrow{PD}$ is perpendicular to the y-axis, $a = m(\angle QPD)$.

So,

$$(*) \qquad \tan a^\circ = \frac{d(\overrightarrow{DQ})}{d(\overrightarrow{PD})} = \frac{|y(Q) - y(P)|}{|x(Q) - x(P)|}.$$

Now, let us suppose that $a > 90$. Then, as you have seen in the Exploration Exercises,

$$(**) \qquad \tan (180 - a)^\circ = \frac{|y(Q) - y(P)|}{|x(Q) - x(P)|}.$$

So, in either case, we can compute a if we know the coordinates of P and Q and know whether $a < 90$ [formula (*)] or $a > 90$ [formula (**)].

However, in the case $a < 90$,

$$\text{if } x(Q) > x(P) \text{ then } y(Q) > y(P)$$

and

$$\text{if } x(Q) < x(P) \text{ then } y(Q) < y(P).$$

So, if $a < 90$ then

$$\frac{y(Q) - y(P)}{x(Q) - x(P)} > 0.$$

Similarly, if $a > 90$ then

$$\frac{y(Q) - y(P)}{x(Q) - x(P)} < 0.$$

The ratio $\frac{y(Q) - y(P)}{x(Q) - x(P)}$ is called <u>the slope</u> of $\overleftrightarrow{PQ}$.

So, to find the inclination of an oblique line, first compute its slope, and then find the measure a of an angle whose tangent is the absolute value of the slope. This number a is the inclination of the line if the slope is positive. If the slope is negative, the inclination is $180 - a$.

Up to now we have been talking about inclinations and slopes of oblique lines. Now, let's consider lines perpendicular to one of the coordinate axes. A line perpendicular to the x-axis has inclination 90 [Why?]. However, since for any two points P and Q on such a line, $x(Q) - x(P) = 0$, the slope of lines perpendicular to the x-axis is not defined [Why?]. On the cther hand, lines perpendicular to the y-axis do not have inclinations [Why?]. But, the slope of such a line is defined to be 0. [Guess why.]

EXERCISES

A. Where possible, find the slope and inclination of the line determined by the given points.

1. A(4, 3) and B(3, 4)
2. C(5, 6) and D(−3, 7)
3. E(0, 4) and F(−4, 3)
4. G(7, 3) and H(3, 3)
5. I(3, 8) and J(3, 9)
6. K(1.32, −1.67) and L(5.67, 1.81)

B. 1. For two lines to be parallel is it sufficient that they have the same inclination? Is it necessary?

2. For two lines to be parallel is it sufficient that they have the same slope? Is it necessary?

3. For two lines to be parallel is it sufficient that they either have the same inclination or have the same slope? Is it necessary?

C. 1. Prove that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ [A(3, 7), B(−5, 9), C(8, 2), and D(0, 4)].

2. Use the idea of slopes to prove that A, B, and C are collinear [A(3, 5), B(2, 6), and C(1, 7)].

3. Given A(2, 5), B(−3, 4), and C(4, 6). Find a point D such that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

4. Given A(3, 8), B(2, 4), and x(C) = 10. Find y(C) such that A, B, and C are collinear.

EQUATIONS OF LINES

Consider, for a point A and a number α such that $0 < \alpha < 180$, the line ℓ through A whose inclination is α.

If $\alpha = 90$, this line is the perpendicular through A to the x-axis. In this case, a point P belongs to ℓ if and only if $x(P) = x(A)$.

If $\alpha \neq 90$, and $P \neq A$ then $P \in \ell$ if and only if

$$\frac{y(P) - y(A)}{x(P) - x(A)} = \text{the slope of } \ell.$$

If the slope of ℓ is the number m, it follows that, whether $P \neq A$ or not,

$$P \in \ell \text{ if and only if } y(P) - y(A) = m[x(P) - x(A)].$$

Thus, we have the following theorem:

> Theorem 9-4.
>
> For each point A, the line through A perpendicular to the x-axis is
>
> $$\{P:\ x(P) = x(A)\}.$$
>
> For each point A, for each number m, the line through A with slope m is
>
> $$\{P:\ y(P) - y(A) = m[x(P) - x(A)]\}.$$

Let's use this theorem. Suppose the coordinates of A are (3, 5) and the slope of the line through A is 2. Then, this line is

$$\{P:\ y(P) - 5 = 2[x(P) - 3]\},$$

or more simply,

$$\{P:\ y(P) = 2 \cdot x(P) - 1\}.$$

To draw a picture of this line, we simply plot points whose coordinates satisfy the equation:

$$(*) \quad y = 2x - 1$$

Some of these are the points with coordinates (1, 1), (5, 9), (−10, −21), and (1000, 1999). Of course, in actually making the picture we would plot only three points--two to determine the line, and the extra one to give us a check on our computing.

The equation (*) is called an equation of the line through A(3, 4) with slope 2.

In general, an equation of the line through $A(x_1, y_1)$ with slope m [with respect to a given coordinate system] is:

$$y - y_1 = m(x - x_1)$$

Given x_1, y_1, and m, the line in question is the set of points whose coordinates (x, y) satisfy this equation [or, of course, any equation equivalent to it.]

An equation of the line through $A(x_1, y_1)$ perpendicular to the x-axis is:

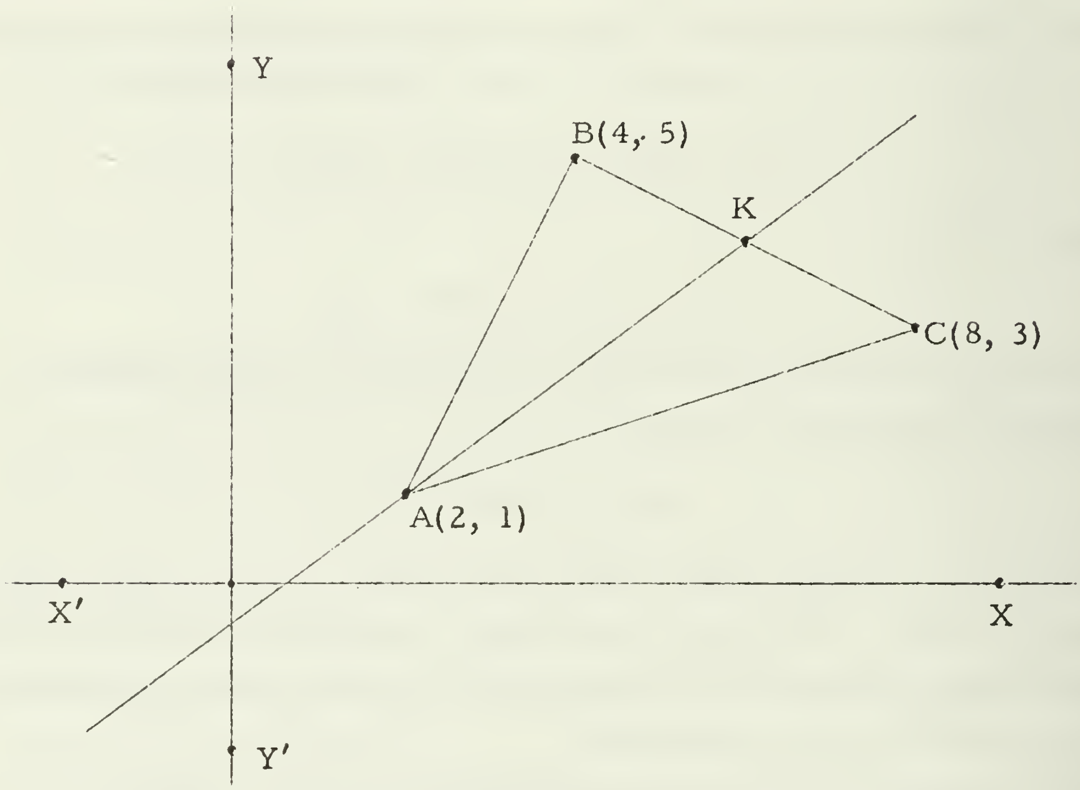
$$x = x_1 \qquad \text{[Explain.]}$$

Example. Find an equation of the line determined by the vertex A and the midpoint K of side $\overleftrightarrow{BC}$ of $\triangle ABC$ [A(2, 1), B(4, 5), and C(8, 3)].

Plan. The line $\overleftrightarrow{AK}$ contains A(2, 1) and K whose coordinates we can determine since it is the midpoint of $\overline{BC}$. So, we can compute the slope of $\overleftrightarrow{AK}$, and use the point-slope equation form:

$$y - y_1 = m(x - x_1)$$

Solution.



Let $x_1 = 2$ and $y_1 = 1$. Now, we want to compute m. Since K is the midpoint of $\overline{BC}$, the coordinates of K are (6, 4). So, the slope of $\overleftrightarrow{AK}$ is

$$\frac{4 - 1}{6 - 2} = \frac{3}{4}.$$

Hence, an equation of $\overleftrightarrow{AK}$ is:

$$y - 1 = \frac{3}{4}(x - 2)$$

A simpler equation of $\overleftrightarrow{AK}$ is:

$$y = \frac{3}{4}x - \frac{1}{2}$$

EXERCISES

A. Find an equation for each of the lines described.

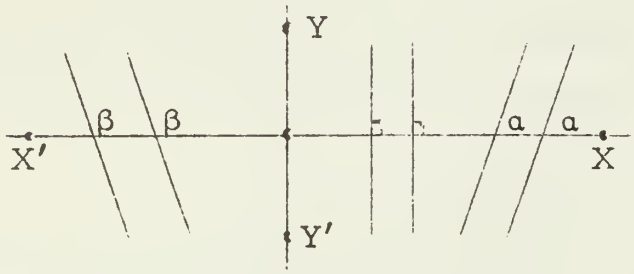
1. It contains the point with coordinates (3, 9) and its slope is 2.
2. It contains the point with coordinates (−5, −3) and its slope is 1.
3. It contains the point with coordinates (5, 7) and it is perpendicular to the x-axis.
4. It contains the points with coordinates (3, 8) and (5, 6).
5. It contains the origin and the point with coordinates (6, 7).
6. It contains the points with coordinates (0, 3) and (4, 0).
7. It passes through the midpoints of $\overline{AB}$ and $\overline{BC}$ of $\triangle ABC$ [A(5, 2), B(7, 5), and C(10, −2)]. What is the slope of $\overleftrightarrow{AC}$?
8. It contains the point with coordinates (7, 3) and is parallel to a line whose slope is $-\frac{1}{3}$.
9. It contains the point with coordinates (2, −5) and is parallel to the line one of whose equations is '$y = 4x - 3$'.
10. It contains the point with coordinates (7, −2) and is parallel to the line one of whose equations is '$3y - 2x = 7$'.
11. Its inclination is 45 and it contains P(7, 11).
12. Its inclination is 90 and it contains the origin.
13. Its inclination is 135 and it contains Q(−5, 3).

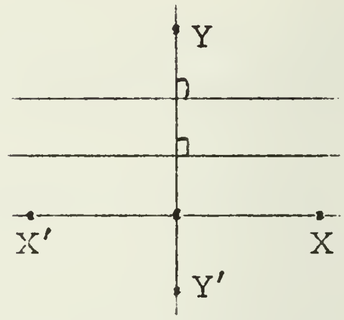
B. Each of the following equations is an equation of a line. Find the slope of the line, and the coordinates of the points in which it intersects the axes.

1. $y - 7 = 3(x - 4)$	2. $y = 3(x - 4) + 7$	3. $y = 3x - 5$
4. $y = 2x + 5$	5. $y = 8x - 16$	6. $y = -\frac{1}{2}x$
7. $y = \frac{1}{3}x - 5$	8. $y = 10x + 6$	9. $3y = 30x + 18$
10. $y - 5x = 4$	11. $y + 7x = 2$	12. $3x - y = 2$
13. $2x + 5y = 10$	14. $6x - 5y = 30$	15. $x + 3y = 9$
16. $\frac{x}{4} + \frac{y}{7} = 1$	17. $\frac{x}{3} - \frac{y}{2} = 1$	18. $\frac{x}{5} - \frac{x}{2} = 1$

PARALLEL LINES AND PERPENDICULAR LINES

Two lines are parallel if and only if they have the same inclination or are both perpendicular to the y-axis.

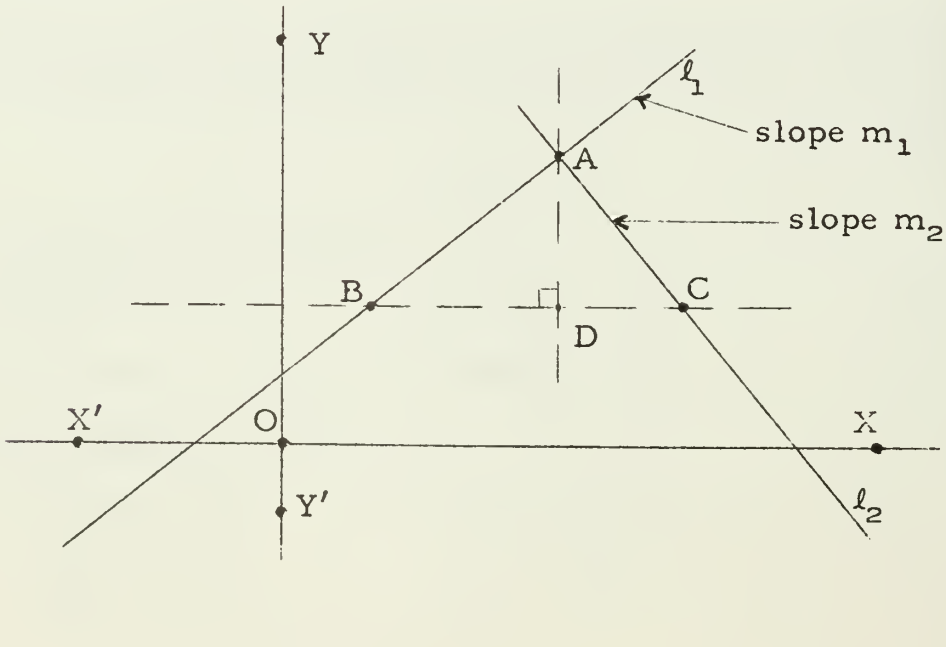




Since lines which are not perpendicular to the x-axis have the same inclination if and only if they have the same slope, and lines are perpendicular to the y-axis if and only if they have slope 0, we have the following theorem:

> Theorem 9-5.
>
> Two lines are parallel if and only if they either have the same slope or are both perpendicular to the x-axis.

So, if each of two lines has a slope, it is easy to tell whether they are parallel. Just compute their slopes and compare. There should be an equally easy way of telling whether two lines which have slopes are perpendicular. To see how, let us consider two lines ℓ_1 and ℓ_2 which have slopes m_1 and m_2, respectively.



Suppose that $\ell_1 \perp \ell_2$ at A. Consider points B and C different from A with $B \in \ell_1$, $C \in \ell_2$, and $\overleftrightarrow{BC}$ perpendicular to the y-axis. Then ΔCAB is right-angled at A. Let D be the foot of the altitude of ΔCAB from A. By an earlier theorem, we know that

$$[d(\overline{AD})]^2 = d(\overline{BD}) \cdot d(\overline{DC}).$$

Now, since

$$|m_1| = \frac{d(\overline{AD})}{d(\overline{BD})} \quad \text{and} \quad |m_2| = \frac{d(\overline{AD})}{d(\overline{DC})}$$

it follows that

$$(1) \qquad |m_1 m_2| = |m_1| \cdot |m_2| = \frac{d(\overline{AD}) \cdot d(\overline{AD})}{d(\overline{BD}) \cdot d(\overline{DC})} = 1.$$

However, one of the lines ℓ_1 and ℓ_2 has inclination less than 90 and the other has inclination greater than 90. So, one of the slopes is positive and the other is negative. Hence,

$$(2) \qquad m_1 m_2 < 0.$$

So, from (1) and (2), it follows that

$$m_1 m_2 = -1.$$

We have shown that if $\ell_1 \perp \ell_2$ then $m_1 m_2 = -1$. Now, let's consider the converse.

Suppose that $m_1 m_2 = -1$. Then, since $|m_1 m_2| = 1$,

$$\frac{d(\overline{AD})}{d(\overline{DB})} \cdot \frac{d(\overline{DA})}{d(\overline{DC})} = 1.$$

So,

$$\frac{d(\overline{AD})}{d(\overline{BD})} = \frac{d(\overline{DC})}{d(\overline{DA})}.$$

Since $\angle ADC \cong \angle BDA$, it follows [from what similarity theorem?] that $ADC \longleftrightarrow BDA$ is a similarity. So, $\angle CAD \cong \angle ABD$. But, in the right triangle ΔADB, $\angle DAB$ and $\angle ABD$ are complementary. So, $\angle CAD$ and $\angle DAB$ are complementary.

Now, since $m_1 m_2 < 0$, the inclination of one line is less than 90 and the inclination of the other is greater than 90. From this it follows that D is in the interior of $\angle CAB$. Hence, because $\angle CAD$ and $\angle DAB$ are complementary, it follows that $\angle CAB$ is a right angle. So, $\ell_1 \perp \ell_2$.

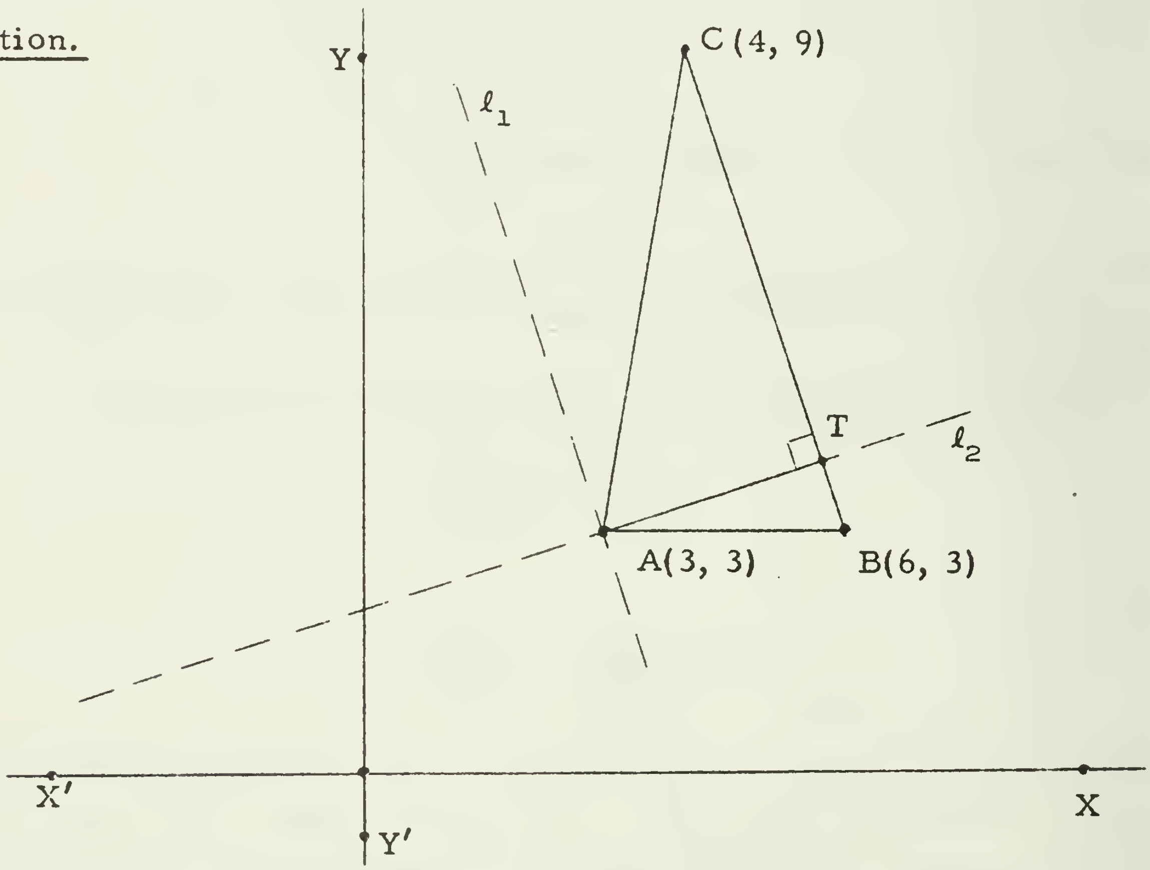
We have shown that if $m_1 m_2 = -1$ then $\ell_1 \perp \ell_2$. Combining this result with the converse proved before, we have the following theorem:

> Theorem 9-6.
>
> Two lines are perpendicular if and only if either the product of their slopes is -1 or one is perpendicular to the x-axis and the other to the y-axis.

[Explain the last part of the theorem.] Sometimes people say that two numbers whose product is -1 are negative reciprocals of each other. Restate the theorem using this expression.

Example 1. Given $\triangle ABC$ as shown. Find an equation of the line ℓ_1 through A parallel to $\overleftrightarrow{BC}$, and an equation of the line ℓ_2 which contains the altitude of $\triangle ABC$ from A.

Solution.



The slope of ℓ_1 must be the slope of $\overleftrightarrow{BC}$, and the slope of ℓ_2 must be the negative reciprocal of the slope of $\overleftrightarrow{BC}$. So, let's begin by finding the slope of $\overleftrightarrow{BC}$.

$$\text{slope of } \overleftrightarrow{BC} = \frac{9 - 3}{4 - 6} = -3$$

Therefore, the slope of ℓ_1 is -3 and the slope of ℓ_2 is $\frac{1}{3}$.

Now, we use the point-slope equation form:

$$y - y_1 = m(x - x_1)$$

For ℓ_1: $y - 3 = -3(x - 3)$	For ℓ_2: $y - 3 = \frac{1}{3}(x - 3)$
$y = -3x + 12$	$y = \frac{1}{3}x + 2$

<u>Example 2.</u> Find the coordinates of the foot of the altitude in Example 1.

<u>Solution.</u> Since T is the point of intersection of ℓ_2 and $\overleftrightarrow{BC}$, it follows that the ordered pair $(x(T), y(T))$ must satisfy equations of ℓ_2 and $\overleftrightarrow{BC}$.

We already have an equation of ℓ_2. So, one of the things we know about $(x(T), y(T))$ is that

(1) $$y(T) = \frac{1}{3} \cdot x(T) + 2\,.$$

Now, let's find an equation of $\overleftrightarrow{BC}$. From Example 1, we know that the slope of $\overleftrightarrow{BC}$ is -3, and, since $C \in \overleftrightarrow{BC}$, an equation of $\overleftrightarrow{BC}$ is:

$$y - 9 = -3(x - 4)$$
$$y = -3x + 21$$

Since $T \in \overleftrightarrow{BC}$, a second thing we know about $(x(T), y(T))$ is that

(2) $$y(T) = -3 \cdot x(T) + 21\,.$$

Substituting from (1) and (2), we get:

$$\frac{1}{3} \cdot x(T) + 2 = -3 \cdot x(T) + 21$$

Solving this last equation:

$$x(T) + 6 = -9 \cdot x(T) + 63$$
$$10 \cdot x(T) = 57$$
$$x(T) = 5.7$$

Substituting from this equation into (2) gives us:

$$y(T) = -3 \cdot 5.7 + 21$$
$$= -17.1 + 21$$
$$= 3.9$$

So, the coordinates of T are (5.7, 3.9).

EXERCISES

A. In each exercise you are given an equation of a line and the coordinates of a point. Find equations of the lines ℓ_1 and ℓ_2 which contain the given point and are parallel and perpendicular, respectively, to the given line, and draw a picture of the three lines, labeling each with its equation.

1. $y = -2x + 5$, $(3, 4)$
2. $3x - y - 6 = 0$, $(2, 1)$
3. $4x + 7y + 14 = 0$, $(-2, -6)$
4. $x + 3 = 0$, $(2, 5)$
5. $y - 2 = 0$, $(4, -7)$
6. $y = x$, $(-5, -6)$

B. Solve these problems.

1. Show that the four points $A(10, 3)$, $B(8, 7)$, $C(0, 3)$ and $D(2, -1)$ are the vertices of a rectangle. [Hint. Use slopes.]

2. Show that the four points $A(1, -5)$, $B(5, 2)$, $C(1, 4)$, and $D(-3, -3)$ are the vertices of a parallelogram, and that the parallelogram is not a rectangle.

3. Show that the four points $A(4, -5)$, $B(7, 4)$, $C(1, 6)$, and $D(-2, -3)$ are not the vertices of a square.

4. Find equations of the lines which contain the altitudes of the triangle ΔABC [$A(3, 2)$, $B(-2, 4)$, and $C(5, -5)$].

5. Find equations of the perpendicular bisectors of the sides of the triangle mentioned in Exercise 4.

6. Find the coordinates of the point of concurrence of the perpendicular bisectors in Exercise 5.

7. Prove by analytic geometry that the diagonals of a parallelogram are perpendicular if and only if the parallelogram is a rhombus.

8. Suppose a line intersects the x-axis at $A(3, 0)$ and the y-axis at $B(0, 4)$. What is the slope of $\overleftrightarrow{AB}$? Give the point-slope form of an equation of $\overleftrightarrow{AB}$. Transform this equation to one of the form: $\frac{x}{?} + \frac{y}{?} = 1$.

9. Repeat Exercise 8 for
   (a) $C(-2, 0)$ and $D(0, 5)$;
   (b) $E(4, 0)$ and $F(0, -6)$;
   (c) $G(-6, 0)$ and $H(0, -2)$;
   (d) $I(a, 0)$ and $J(0, b)$, [$ab \neq 0$]

SUMMARY OF SECTION 6.09

Notation and terminology

Theorems

9-1. For each point P, for each point Q,

$d(\overleftrightarrow{PQ}) = \sqrt{(|x(Q) - x(P)|)^2 + (|y(Q) - y(P)|)^2}$. [The distance formula]

9-2. $\forall_P \forall_Q \forall_M$ M is the midpoint of $\overleftrightarrow{PQ}$ if and only if

$x(M) = \frac{x(P) + x(Q)}{2}$ and $y(M) = \frac{y(P) + y(Q)}{2}$. [The midpoint formula]

9-3. The medians of a triangle are concurrent at a point which divides each median from the vertex to the midpoint of the opposite side in the ratio 2 to 1.

9-4. For each point A, the line through A perpendicular to the x-axis is $\{P: x(P) = x(A)\}$. For each point A, for each number m, the line through A with slope m is $\{P: y(P) - y(A) = m[x(P) - x(A)]\}$.

9-5. Two lines are parallel if and only if they either have the same slope or are both perpendicular to the x-axis.

9-6. Two lines are perpendicular if and only if either the product of their slopes is -1 or one is perpendicular to the x-axis and the other to the y-axis.

6-10 Circles.--A circle, like any geometric figure, is a set of points.

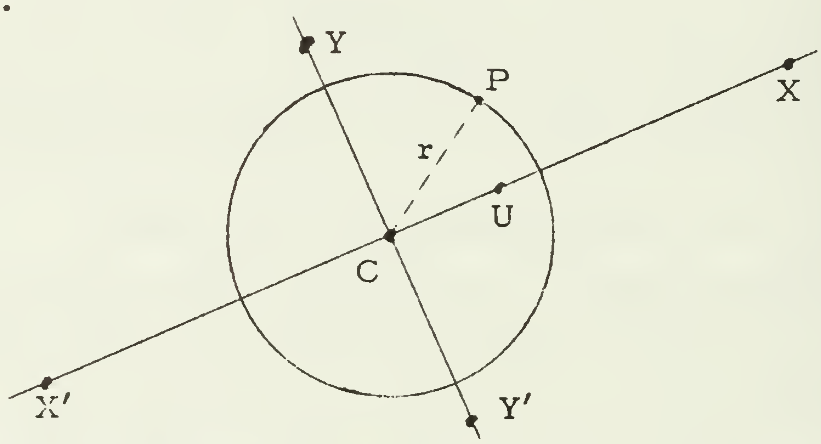
For each point C and each number $r > 0$, the circle with center C and radius r is the set of all points whose distance from C is r. [The set which consists of just the single point C is sometimes called a degenerate circle, or a point-circle. Its radius is 0.]

So, the circle with center C and radius r is $\{P: CP = r\}$.

EQUATION OF A CIRCLE

Some properties of circles are most easily developed by analytic methods, particularly those relating to how circles intersect. Before going into this matter formally, experiment by drawing pairs of circles and seeing what can happen.

In order to study these properties analytically, we must first learn how to describe a circle with reference to a coordinate system. The simplest situation is one in which the origin of the system is the center of the circle.



By definition, a point P belongs to the circle if and only if its distance from the origin is r. So, by the distance formula, P belongs to the circle if and only if

$$\sqrt{[x(P) - 0]^2 + [y(P)) - 0]^2} = r.$$

That is, if and only if

$$[x(P)]^2 + [y(P)]^2 = r^2.$$

We shall say, for $r > 0$, that the equation:

$$(*) \qquad x^2 + y^2 = r^2$$

is an equation of the circle with origin as center and radius r. Thus, if a point belongs to this circle then its coordinates satisfy (*). And, if the coordinates of a point satisfy (*) then the point belongs to this circle.

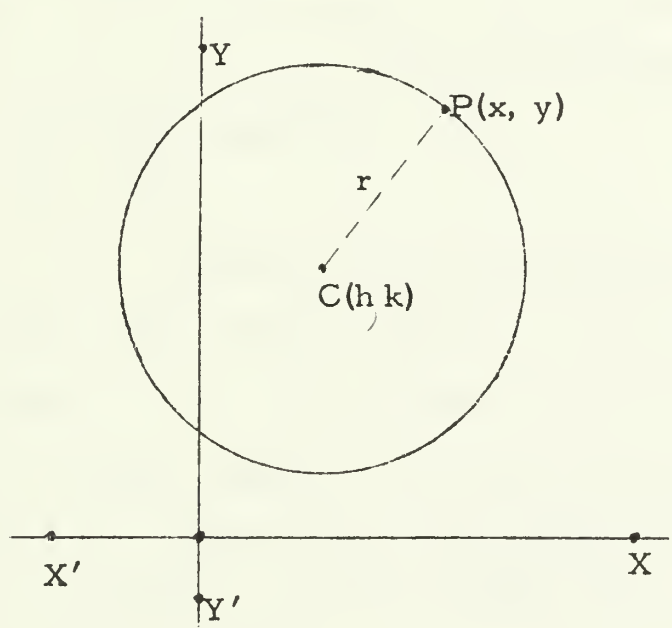
(1) Find an equation of a circle with center O and radius 7.

(2) What is the radius and what are the coordinates of the center of a circle one of whose equations is '$x^2 + y^2 = 100$'?

(3) Repeat (2) for the equation '$x^2 + y^2 = 2$'.

(4) Find an equation of the circle with center C and radius 2 with respect to a coordinate system for which the coordinates of C are (3, 5).

If you were able to solve problem (4), then you probably know the general procedure for writing an equation for the circle with center C and radius r with respect to a coordinate system for which the coordinates of C are (h, k).



A point P belongs to the circle if and only if its distance from C is r. So, P belongs to the circle if and only if

$$\sqrt{[x(P) - h]^2 + [y(P) - k]^2} = r.$$

That is, if and only if

$$[x(P) - h]^2 + [y(P) - k]^2 = r^2.$$

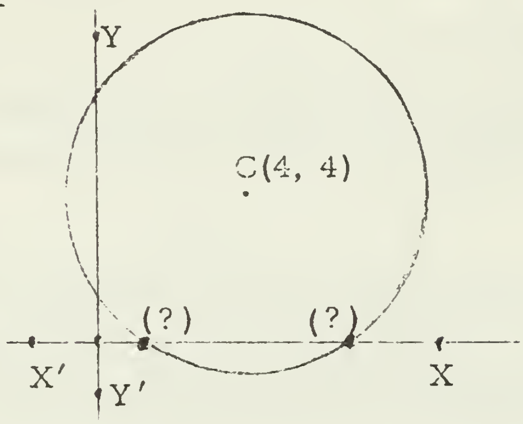
So, for $r > 0$, the equation:

$$(**) \qquad (x - h)^2 + (y - k)^2 = r^2$$

is <u>an</u> <u>equation</u> <u>of</u> <u>the</u> <u>circle</u> whose center is the point C(h, k) and whose radius is r [with respect to the appropriate coordinate system].

<u>Example.</u> Find the coordinates of the points in the intersection of the x-axis and the circle with center (4, 4) and radius 5.

<u>Solution.</u>



<u>Equation of the circle:</u>

$$(x - 4)^2 + (y - 4)^2 = 25$$

<u>Equation of the x-axis:</u>

$$y = 0$$

Now, P is a point in the intersection of the x-axis and the circle if and only if (x(P), y(P)) satisfies both equations. That is, P is a point in the intersection if and only if

$$(1) \qquad [x(P) - 4]^2 + [y(P) - 4]^2 = 25$$

and

$$(2) \qquad y(P) = 0.$$

Substituting from (2) into (1), we get:

$$[x(P) - 4]^2 + [0 - 4]^2 = 25$$

$$[x(P) - 4]^2 + 16 = 25$$

$$[x(P) - 4]^2 = 9$$

$$x(P) - 4 = 3 \quad \text{or} \quad x(P) - 4 = -3$$

$$x(P) = 7 \quad \text{or} \quad x(P) = 1$$

So, if P is a point in the intersection then its coordinates are (7, 0) or (1, 0). Do each of these ordered pairs satisfy (1) and (2)? Clearly, each one satisfies (2). Let's see if each one satisfies:

$$(1) \qquad [x(P) - 4]^2 + [y(P) - 4]^2 = 25.$$

$(7 - 4)^2 + (0 - 4)^2$	25		$(1 - 4)^2 + (0 - 4)^2$	25
$3^2 + (-4)^2$	25		$(-3)^2 + (-4)^2$	25
$9 + 16$	25		$9 + 16$	25
$25 =$	25 ✓		$25 =$	25 ✓

So, there are two points in the intersection and their coordinates are (1, 0), and (7, 0).

EXERCISES

A. Write an equation for each of the circles described.

1. center $C(2, 1)$ and radius 3
2. center $C(1, 2)$ and radius 3
3. center $C(-5, 6)$ and radius $\sqrt{5}$
4. center $C(-3, -5)$ and radius 11
5. center is the midpoint of $\overleftrightarrow{AB}$ [$A(3, 2)$ and $B(7, 12)$] and A is a point of the circle
6. center is the midpoint of $\overleftrightarrow{AB}$ [$A(6, 0)$ and $B(0, 8)$] and the circle contains the origin

B. For each circle described by an equation, give the coordinates of its center, and give its radius.

1. $x^2 + y^2 = 81$
2. $y^2 + x^2 = 144$
3. $(x - 1)^2 + (y - 2)^2 = 169$
4. $(x + 1)^2 + (y - 2)^2 = 225$
5. $(x - 3)^2 + y^2 = 100$
6. $(x + 3)^2 + (y + 4)^2 = 1$
7. $x^2 + (y + 9)^2 = 9$
8. $(x - \sqrt{2})^2 + (y + \sqrt{3})^2 = 5$
9. $\{P: x(P) > 0 \text{ and } y(P) > 0\}$ is called <u>the first quadrant</u>.
   $\{P: x(P) < 0 \text{ and } y(P) > 0\}$ is called <u>the second quadrant</u>.
   $\{P: x(P) < 0 \text{ and } y(P) < 0\}$ is called <u>the third quadrant</u>.
   What is <u>the fourth quadrant</u>? Which of the circles in Exercises 1-8 are subsets of Quadrant III?

C. Find the coordinates of the points in the intersection of the given sets.

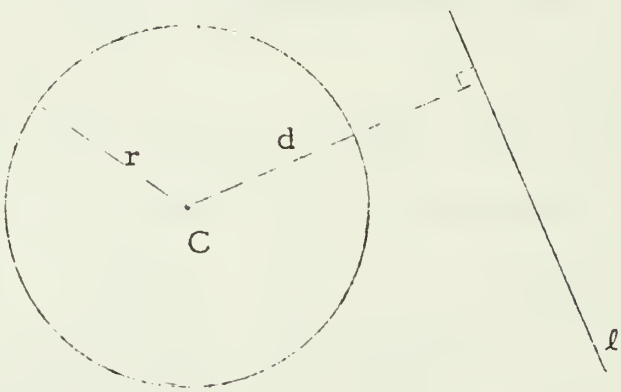
1. circle: $x^2 + y^2 = 25$; line: $x = 3$
2. circle: $x^2 + y^2 = 25$; line: $x = 4$
3. circle: $x^2 + y^2 = 25$; line: $x = 5$
4. circle: $x^2 + y^2 = 25$; line: $x = 10$
5. circle: $x^2 + y^2 = 25$; line: $y = \frac{4}{3}x$
6. circle: $(x - 3)^2 + (y - 4)^2 = 25$; line: $x = 0$

INTERSECTIONS OF CIRCLES AND LINES

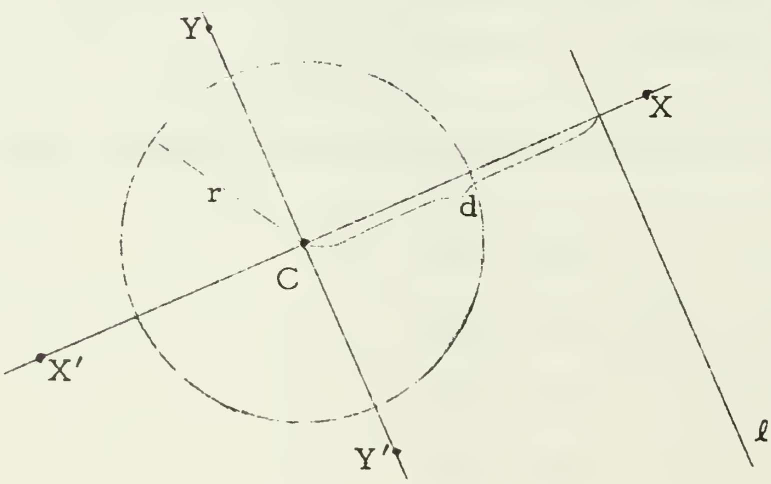
There seem to be three ways in which a line can be related to a circle.

empty intersection 1-point intersection 2-point intersection

Let's prove that this is the case. Consider a circle with center C and radius r, and a line ℓ whose distance from C is d.



Set up a coordinate system with origin C and the perpendicular through C to ℓ as x-axis. Choose the positive direction on the x-axis



so that the equation of ℓ is '$x = d$'. With respect to this coordinate system, the equation of the circle is '$x^2 + y^2 = r^2$'.

At this point, our intuition tells us that there will be two points of intersection if $d < r$. How many if $d = r$? How many if $d > r$?

This is easy to check. A point P belongs to the intersection if and only if

(1) $$[x(P)]^2 + [y(P)]^2 = r^2$$

and (2) $$x(P) = d.$$

Substituting from (2) into (1), we obtain:

$$d^2 + [y(P)]^2 = r^2$$

(3) $$[y(P)]^2 = r^2 - d^2$$

If there is a point P which satisfies (3), then $r^2 - d^2 \geq 0$. [Why?]

If $r^2 - d^2 > 0$ then (3) is equivalent to:

(4) $$y(P) = \sqrt{r^2 - d^2} \text{ or } y(P) = -\sqrt{r^2 - d^2}$$

By substitution, we find that the points whose coordinates are $(d, \sqrt{r^2 - d^2})$ and $(d, -\sqrt{r^2 - d^2})$ satisfy (1) and (2). So, if $r^2 - d^2 > 0$--that is, if $d < r$--then the line intersects the circle in two points.

On the other hand, if $r^2 - d^2 = 0$--that is, if $d = r$--then the line intersects the circle in the single point whose coordinates are $(d, 0)$

Finally, as indicated above, if $r^2 - d^2 \not\geq 0$--that is, if $d > r$--then there is no point P which belongs to both the circle and the line.

In summary, given a circle with radius r and a line ℓ such that the distance between the center and ℓ is d,

(1) if $d < r$ then the intersection contains exactly two points,

(2) if $d = r$ then the intersection contains exactly one point,

and (3) if $d > r$ then the intersection contains no points.

Suppose we are told that a line intersects a circle in two points. Let's see what we can say about the distance between this line and the center. Is the distance equal to the radius? Why not? Is the distance greater than the radius? Why not?

Show that if a line intersects a circle in exactly one point then the distance between the center and the line is equal to the radius of the circle.

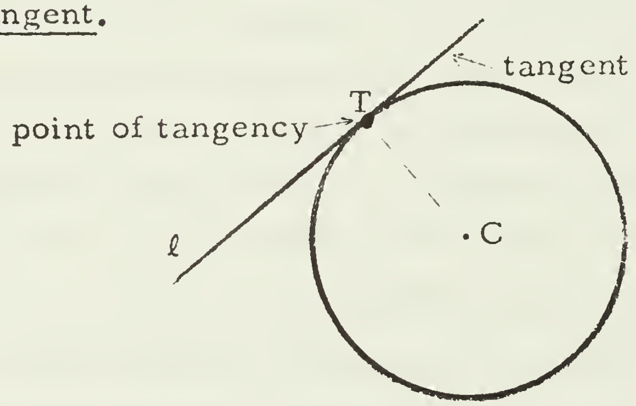
Show that if a line does not intersect a circle then the distance between the center and the line is greater than the radius.

Thus, we have the following theorem:

> Theorem 10-1.
>
> For each circle c with radius r and each line ℓ at a distance d from the center of c,
>
> $\ell \cap c$ consists of two points if and only if $d < r$,
>
> $\ell \cap c$ consists of one point if and only if $d = r$, and
>
> $\ell \cap c = \emptyset$ if and only if $d > r$.

TANGENT, SECANTS, and CHORDS

A line which intersects a circle in exactly one point is said to be tangent to the circle, and the circle is said to be tangent to the line. The point of intersection is called the point of tangency, and the line is called a tangent.



What is the distance between a tangent ℓ to the circle at T and the center of the circle? Since, by the definition of a tangent, ℓ intersects the circle in exactly one point it follows from Theorem 10-1 that the distance is equal to the radius.

Now, consider the segment $\overline{CT}$. Since the measure of $\overline{CT}$ is the distance between C and ℓ and, since this is the measure of the perpendicular segment from C to ℓ, it follows that $\ell \perp \overline{CT}$. So, we have shown that if ℓ is tangent to the circle at T then $\ell \perp \overline{CT}$.

Conversely, suppose ℓ is the line perpendicular to $\overline{CT}$ at T. Is ℓ a tangent to the circle? Yes, because since CT is the distance between ℓ and C and $CT = r$, it follows from Theorem 10-1 that the

intersection of ℓ and the circle contains exactly one point. So, by definition, ℓ is tangent to the circle.

A segment which joins the center of a circle to one of its points is called a radial segment. The measure of each such segment is the radius of the circle. For short, one often calls a radial segment a radius.

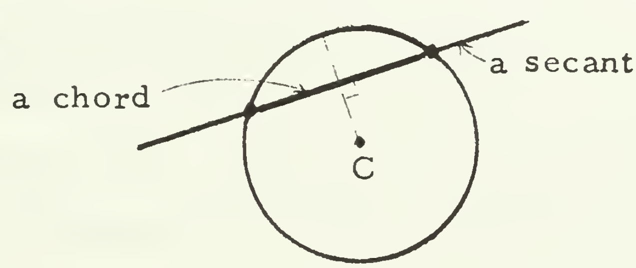
So, we have the following theorem:

> Theorem 10-2.
>
> A line is tangent to a circle at a given point if and only if it contains the point and is perpendicular to the radius at that point.

How many tangents are there to a circle through a point P (a) if P is on the circle; (b) if P is "inside" the circle; (c) if P is "outside" the circle?

Consider a line which intersects a circle in two points. Such a line is called a secant of the circle. [Compare the radius with the distance between the center and a secant.]

The points at which a secant intersects a circle are the end points



of a segment, a segment which is called a chord of the circle. As we have shown [see sentence (4) on page 6-275], the perpendicular through C to a chord bisects the chord. This gives us the following theorem:

> Theorem 10-3.
>
> The radius perpendicular to a chord bisects the chord.

A chord which contains the center of a circle is called a diameter of the circle.

EXERCISES

A. 1. Draw a circle with center C. Mark a point T on the circle.

2. Using compass and straight-edge only, draw the line tangent to the circle at T.

B. 1. Draw a circle with center C, and draw a secant to the circle. Let A and B be the points of intersection.

2. Using compass and straight-edge only, draw the lines tangent to the circle at A and B, respectively.

C. 1. Draw ΔABC. Locate the point which is equidistant from the three vertices.

2. Draw a circle which contains all three vertices of ΔABC. How many such circles are there?

D. 1. Draw ΔABC. Locate the point of concurrence of the angle bisectors.

2. Draw a circle which is tangent to each of the three lines which contain the sides of the triangle and whose center is in the interior of the triangle.

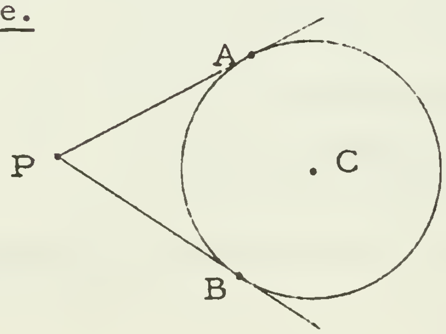
E. 1. Suppose ℓ is the tangent at T to a circle with center C. Let S be a point of ℓ such that ST = 12. If the radius is 5, what is SC?

2. A chord of a circle has measure 8. If the radius is 5, what is the distance between the center and the chord?

3. The distance between the center of a circle and one of its chords is 5. If the radius is 13, what is the measure of the chord?

* * *

Example.



Hypothesis: $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangents to the circle with center C at the points A and B, respectively

Conclusion: PA = PB

Plan. Consider the triangles $\triangle ACP$ and $\triangle BCP$. Try to show that there is a congruence for which $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ are corresponding sides. The triangles are right triangles with right angles at A and B. Also, CA = CB and CP = CP. So, use the hypotenuse-leg theorem to show that ACP ⟷ BCP is a congruence.

Solution I. [column proof]

(1)	A and B belong to a circle with center C	[Hypothesis]
(2)	CA = CB	[(1); def. of circle]
(3)	$\overleftrightarrow{PA}$ is the tangent to the circle at A	[Hypothesis]
(4)	A line is tangent to a circle at a given point if and only if it contains the point and is perpendicular to the radius at that point.	[Theorem 10-2]
(5)	$\overleftrightarrow{CA} \perp \overleftrightarrow{PA}$	[(3) and the only-if-part of (4)]
(6)	Each of the four angles contained in two perpendicular lines is a right angle.	[theorem]
(7)	$\angle CAP$ is a right angle	[(5) and (6)]
(8)	$\angle CBP$ is a right angle	[steps like (3) - (7)]
(9)	CP = CP	[Identity]
(10)	A, C, P and B, C, P are vertices of right triangles with right angles at A and B, respectively	[(7) and (8); definition of a right triangle]
(11)	h.ℓ.	[theorem]
(12)	ACP ⟷ BCP is a congruence	[(10), (9), (2), and (11)]
(13)	PA = PB	[(12); def. of congruence]

Solution II. [paragraph proof]

Since $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangents to the circle with center C at the points A and B, respectively, it follows that $\overline{CA} \perp \overline{PA}$ and $\overline{CB} \perp \overline{PB}$ (1). So, $\triangle ACP$ and $\triangle BCP$ are right-angled at A and B, respectively (2). Now, by the definition of a circle, the legs $\overline{CA}$ and $\overline{CB}$ are congruent. Also, the triangles have the same hypotenuse. So, by h.ℓ., $ACP \longleftrightarrow BCP$ is a congruence. Therefore, PA = PB.

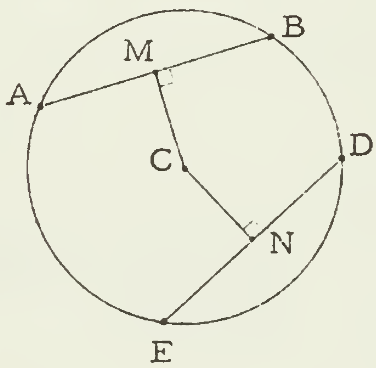
---

(1) A line is tangent to a circle at a given point if and only if it contains the point and is perpendicular to the radius at that point.

(2) Each of the four angles contained in two perpendicular lines is a right angle.

* * *

F. 1.



Hypothesis: AB = DE

Conclusion: CM = CN

2. Interchange the Hypothesis and Conclusion of Exercise 1.

3. Suppose that $\overline{AB}$ is a diameter of a circle and that ℓ and m are tangents to the circle at A and B, respectively. Show that $\ell \parallel m$.

4. Suppose that $\ell \parallel m$ and that ℓ and m are tangents to a circle at T and S, respectively. Show that $\overline{TS}$ is a diameter.

5. Show that the perpendicular bisector of a chord contains the center of the circle.

6. Show that the line determined by the center of a circle and the midpoint of a chord which is not a diameter is perpendicular to the chord.

7. Prove that the diameters of a circle are congruent.

* * *

The preceding exercises give us several important theorems:

<u>Theorem 10-4.</u>

Two chords of a circle are congruent if and only if they are equidistant from the center.

<u>Theorem 10-5.</u>

Two tangents to a circle are parallel if and only if the points of tangency are the end points of a diameter.

<u>Theorem 10-6.</u>

The perpendicular bisector of a chord contains the center of the circle.

<u>Theorem 10-7.</u>

The line containing the center of a circle and the midpoint of a chord of the circle which is not a diameter is perpendicular to the chord.

<u>Theorem 10-8.</u>

The segments joining the point of intersection of two tangents to the points of tangency are congruent.

<u>Theorem 10-9.</u>

The diameters of a circle are congruent.

CIRCUMCIRCLE AND INCIRCLE OF A TRIANGLE

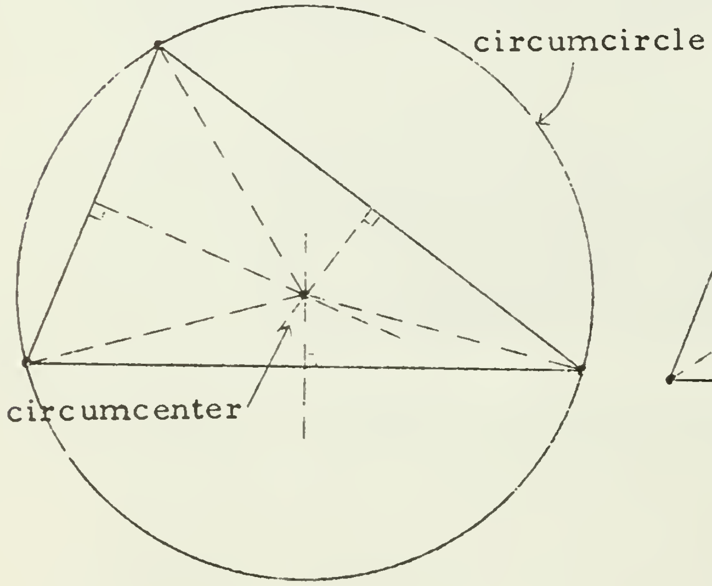
You can conclude from Theorems 3-3 and 4-19 that the point of concurrence P of the perpendicular bisectors of the sides of ΔABC is equidistant from the vertices of the triangle. So, a circle with center P which contains A must also contain B and C. Such a circle is called a circumcircle [its center is called a circumcenter] of the triangle. [See figure below].

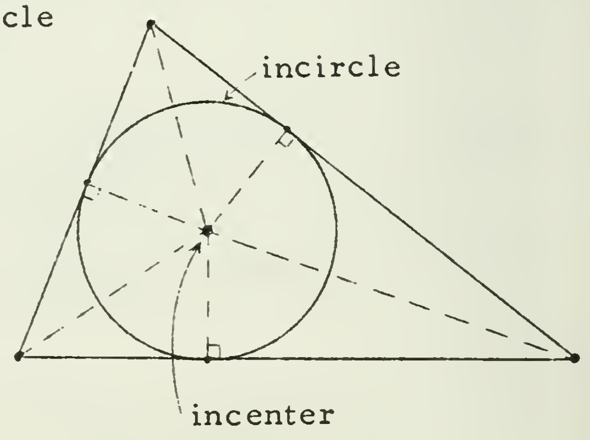
Since there is only one point which is equidistant from the vertices of a triangle, a triangle has only one circumcircle and only one circumcenter. Explain how this leads to the following theorem:

> Theorem 10-10.
> Three noncollinear points determine a circle.

You also proved in an earlier section [page 6-134] that the bisectors of the angles of a triangle are concurrent. Hence, for each triangle, there is a unique point in its interior which is equidistant from the lines containing the sides of the triangle. This unique point is called the incenter of the triangle and is the center of the circle which is tangent to the lines containing the sides of the triangle. This circle is called the incircle of the triangle. [See figure below].

> Theorem 10-11.
> Each triangle has a unique incircle.





EXERCISES

A. Mark three noncollinear points M, N, and R. Using compass and straight-edge only, locate the center of the circle which contains these three points.

B. Here is a part of a circle. Using compass and straight-edge only, locate the center of the circle and draw the rest of it.

C. Draw rectangle ABCD. Now draw the circumcircle of $\triangle ABC$. Does this circumcircle contain the point D? Prove that it should.

D. 1. Draw $\triangle ABC$ and draw its incircle.

2. If M, N, and Q are the points of tangency, and $M \in \overline{AB}$, $N \in \overline{BC}$, and $Q \in \overline{CA}$, show that AM + BN + CQ is half the perimeter of $\triangle ABC$.

E. 1. Compute the radius of the circumcircle of an equilateral triangle whose side measure is s.

2. Compute the radius of the incircle of the equilateral triangle in Exercise 1.

3. Show that the incircle and the circumcircle of an equilateral triangle are concentric--that is, that they have the same center.

☆F. 1. Draw $\triangle ABC$, and draw the bisectors of the exterior angles of the triangle.

2. Each two of the lines containing the bisectors intersect in a point called an excenter of the triangle.

3. Draw all the excircles [or: the escribed circles] of $\triangle ABC$. How many are there?

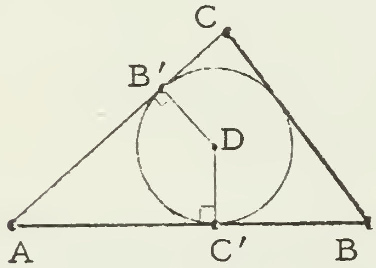
* * *

The point of concurrence of the altitudes of a triangle has a special name. It is called <u>the orthocenter</u> of the triangle. Look at Exercise 3 on page 6-167. Can you describe the special circle whose center is the orthocenter of the triangle?

The medians of a triangle are concurrent. They meet in a point called <u>the centroid</u> [or: center of gravity] of the triangle. If you cut a triangle out of cardboard, you can balance the triangle on your finger-tip by supporting the triangle at its centroid.

* * *

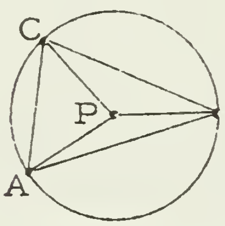
<u>G.</u> 1.



<u>Hypothesis:</u> D is the incenter of ΔABC

<u>Conclusion:</u> $\angle A$ and $\angle D$ are supplementary

2.



<u>Hypothesis:</u> the circumcenter P is in the interior of ΔABC

<u>Conclusion:</u> $m(\angle ACB) = \frac{1}{2} \cdot m(\angle APB)$

[<u>Hint.</u> Let $\{E\} = \overrightarrow{CP} \cap \overline{AB}$.]

3. Suppose the circumcenter of a triangle belongs to one of the sides of the triangle. Prove that the triangle is a right triangle.

☆ 4. Suppose M_1, M_2, and M_3 are the midpoints of the three sides of a triangle. Prove that the circumcenter of the triangle is the orthocenter of $\Delta M_1M_2M_3$.

EXPLORATION EXERCISES

<u>A.</u> Imagine that you have two hoops made of very thin but sturdy wire. [In fact, pretend that the wire has no thickness at all.] Suppose that the radius of one hoop is r and the radius of the other hoop is s, and that $r \geq s$. Pretend that you have tossed the hoops into the air and they have fallen onto a plane surface. The distance between their centers is d.

1. If $d > r + s$, what can you say about the hoops? That is, tell the relative positions of the hoops.

2. If $d = r + s$, what can you say about the hoops?

3. Suppose $d = 0$. Could the hoops be touching in this case? If so, at how many points? Must the hoops be touching?

4. Suppose $d < r + s$. Could the hoops be touching in this case? Must the hoops be touching? [What if $r = 10$, $s = 5$ and $d = 4$?]

5. Suppose $d > r - s$. Could the hoops be touching in this case? Must they be touching?

6. Suppose $r - s < d < r + s$. Could the hoops be touching in this case? Must they be touching?

B. Now, let's reverse the problem of Part A. This time predict d from a knowledge of the relative positions of the hoops. [Assume that $r \geq s$.]

1. One hoop is outside the other but not touching. [So, d ________.]

2. One hoop is outside the other but touching.

3. One hoop is inside the other but touching.

4. One hoop is inside the other but not touching.

5. The hoops touch at three points.

6. The relative positions of the hoops are different from anything mentioned in Exercises 1-5.

INTERSECTIONS OF CIRCLES

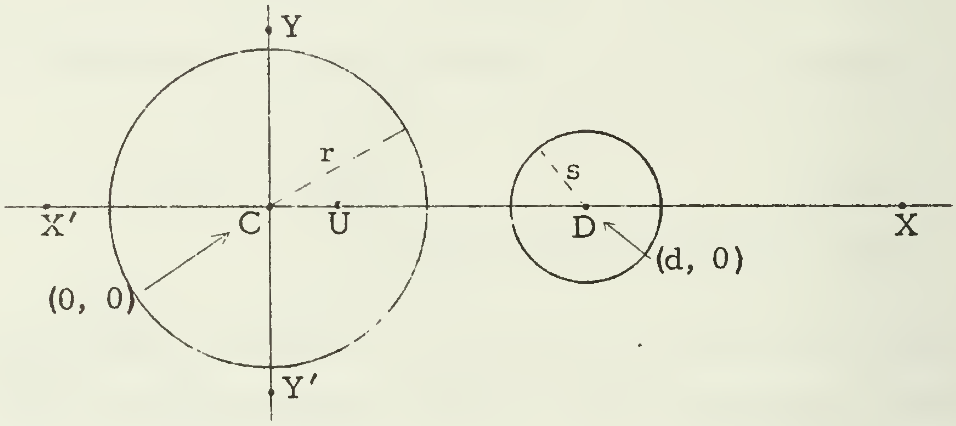
As in the problem of the intersection of a circle and a line, we shall use analytic methods to investigate intersecting circles. In fact, the intersecting circles problem reduces to the circle-line problem.

In the problem of the circle and the line, we discovered how to use the radius (r) and the distance (d) between the line and the circle to tell how many points there are in their intersection. And, conversely, we discovered how to use information about the number of points in the intersection to tell how r and d are related.

In the circle-circle problem, we shall try to make similar discoveries about how the radii of the circles and the distance between their centers is related to the number of points in their intersection.

Consider two circles with centers C and D, and radii r and s, respectively. Let d be the measure of $\overline{CD}$. Assume that $r \geq s$.

Now, set up a coordinate system such that C is the origin, $\overleftrightarrow{CD}$ [the line of centers] is the x-axis, and the x-coordinate of D is non-negative.



With reference to the coordinate system, equations of the circles are:

(1) $$x^2 + y^2 = r^2$$

and:

(2) $$(x - d)^2 + y^2 = s^2$$

Now, a point P belongs to both circles if and only if the coordinates of P satisfy both (1) and (2). Let's try to simplify this condition. Each solution (x, y) of (1) and (2) also satisfies:

(3) $$[x^2 + y^2] - [(x - d)^2 + y^2] = r^2 - s^2$$ [Explain.]

For the same reason ["subtracting (3) from (1)"], each solution (x, y) of (1) and (3) also satisfies (2). So, the coordinates of a point P satisfy (1) and (2) if and only if they satisfy (1) and (3). Hence, P belongs to both circles if and only if its coordinates satisfy (1) and (3).

Let's simplify equation (3).

$$[x^2 + y^2] - [(x^2 - 2dx + d^2) + y^2] = r^2 - s^2$$

$$[x^2 + y^2] - [(x^2 + y^2) - (2dx - d^2)] = r^2 - s^2$$

$$2dx - d^2 = r^2 - s^2$$

$$2d \cdot x = r^2 - s^2 + d^2$$

If $d = 0$, this last equation has no solutions unless $r = s$. Hence, if $d = 0$ and $r \neq s$ then the two circles do not intersect. [What happens if $d = 0$ and $r = s$?]

From now on, suppose that $d \neq 0$. Then, as we have seen, P belongs to both circles if and only if

(1') $\qquad [x(P)]^2 + [y(P)]^2 = r^2$

and (2') $\qquad x(P) = \dfrac{r^2 - s^2 + d^2}{2d}$.

(1') is an equation of the circle with center at the origin and radius r. Equation (2') is an equation of a line perpendicular to the x-axis--that is, to the line of centers $\overleftrightarrow{CD}$. Also, since $r \geq s$, $\dfrac{r^2 - s^2 + d^2}{2d} \geq 0$. So, we are back to our circle-line problem, where the circle has radius r and the distance between its center and the line is $\dfrac{r^2 - s^2 + d^2}{2d}$.

Let's see what we have. If $d \neq 0$, P belongs to both circles if and only if the coordinates of P satisfy (1') and (2'). That is, P belongs to both circles if and only if P belongs to the circle (1') and to the line (2'). From Theorem 10-1, we know about the intersection of a circle and a line. So, if $d \neq 0$, the two circles intersect

in two points	if and only if	$\dfrac{r^2 - s^2 + d^2}{2d} < r$,
in one point	if and only if	$\dfrac{r^2 - s^2 + d^2}{2d} = r$,
not at all	if and only if	$\dfrac{r^2 - s^2 + d^2}{2d} > r$.

Let's simplify the first of these conditions:

$$\frac{r^2 - s^2 + d^2}{2d} < r$$

$$r^2 - s^2 + d^2 < 2rd$$

$$r^2 - 2rd + d^2 - s^2 < 0$$

$$(r - d)^2 - s^2 < 0$$

$$[(r - d) - s][(r - d) + s] < 0$$

$$[(r - s) - d][(r + s) - d] < 0$$

$$[(r - s) - d < 0 \text{ and } (r + s) - d > 0] \text{ or } [(r - s) - d > 0 \text{ and } (r + s) - d < 0]$$

$$[r - s < d < r + s] \text{ or } \underbrace{[r - s > d > r + s]}_{[\text{this can happen only if } s < 0]}$$

So, the circles intersect in two points

if and only if $\qquad r - s < d < r + s \qquad$ [two points].

The same sort of argument, applied to the second condition tells us that the circles intersect in a single point

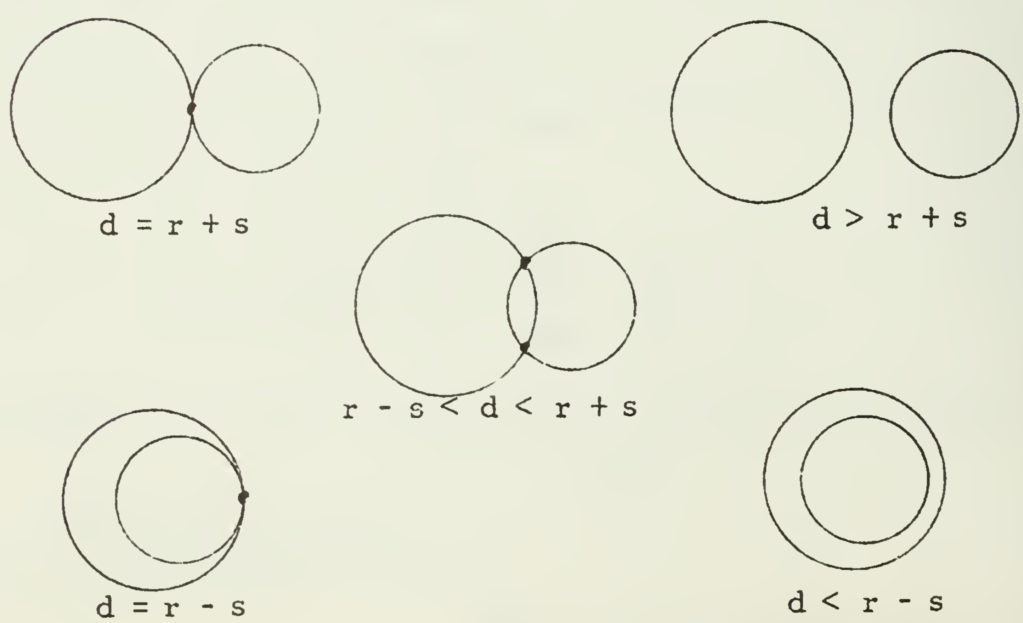
if and only if $\quad d = r - s \;$ or $\; d = r + s \quad$ [one point].

Similarly, the two circles do not intersect

if and only if $\quad d < r - s \;$ or $\; d > r + s \quad$ [no points].

Thus, we have the following theorem:

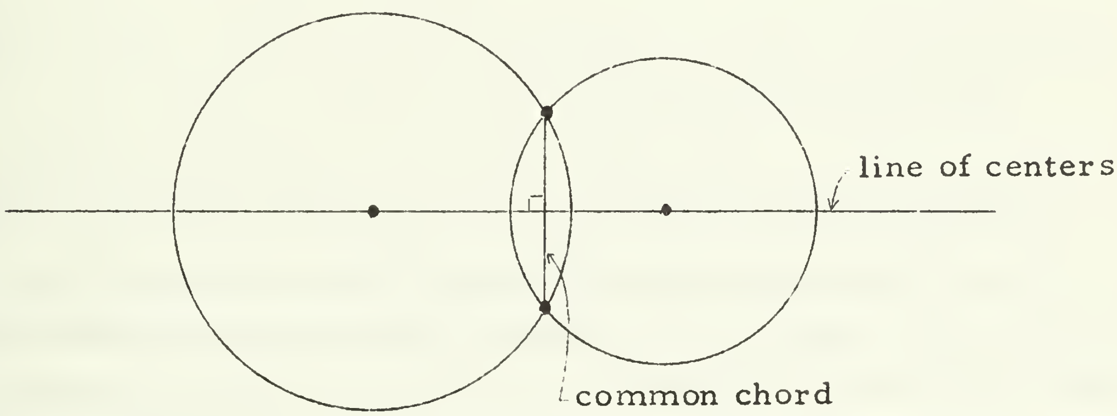
> <u>Theorem 10-12.</u>
>
> For each two circles c_1 and c_2 with radii r and s, respectively, such that $r \geq s$, and with centers a distance $d \neq 0$ apart,
>
> $c_1 \cap c_2$ consists of two points
> if and only if
> $r - s < d < r + s$,
>
> $c_1 \cap c_2$ consists of one point
> if and only if
> $d = r - s$ or $d = r + s$,
>
> and, $c_1 \cap c_2$ is the empty set
> if and only if
> $d < r - s$ or $d > r + s$.



There are other theorems which follow directly from our proof of Theorem 10-12. Do you see that, in the two-point case, equation:

$$(2') \qquad x(P) = \frac{r^2 - s^2 + d^2}{2d}$$

is an equation of the line determined by the two points of intersection of the circles? Since this line is perpendicular to the line of centers, it follows that the line of centers is the perpendicular bisector of the segment joining the two points of intersection. [Why?]



This segment is called <u>the common chord</u> of the two circles. So we proved:

<u>Theorem 10-13.</u> If two circles intersect in two points, the line of centers is the perpendicular bisector of the common chord.

We also proved:

<u>Theorem 10-14.</u> For all nonzero numbers of arithmetic, x, y, and z, if $x \geq y$ then there is a triangle whose side measures are x, y, and z, respectively, if and only if $x - y < z < x + y$.

COMMON TANGENTS

A line tangent to each of two circles is called a common tangent of the circles. If the circles are on the same side of the common tangent, it is called a common external tangent. If the circles are on opposite sides, it is called a common internal tangent.

a common external tangent a common internal tangent

If two circles are tangent to the same line at the same point, they are said to be tangent to each other. If the circles are on the same side of the common tangent they are said to be tangent internally; if they are on opposite sides, they are said to be tangent externally.

circles tangent internally circles tangent externally

EXERCISES

A. 1. Suppose that a line ℓ is a common external tangent to two circles and that ℓ is parallel to the line of centers. What can you say about the radii of the two circles?

2. (a) Suppose that a line ℓ is a common external tangent to two circles, one with radius 3 and the other with radius 8. If the distance between their centers is 13, what is the distance between the points of tangency?

(b) Suppose that the line of centers of these circles intersects the common external tangent in a point P. What are the distances between P and the points of tangency?

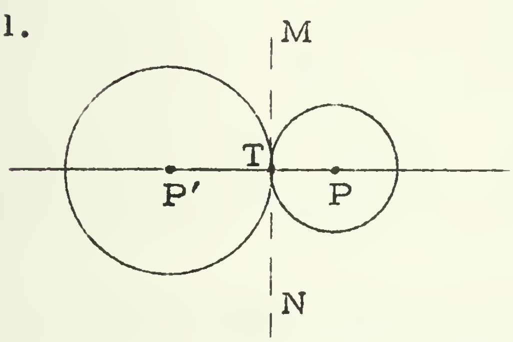
(c) Suppose that a line m is a common internal tangent to these circles. What is the distance between the points of tangency?

3. Let ℓ be a common internal tangent to two circles with the same radius. Show that the segment joining the points of tangency is bisected by the line of centers.

4. (a) Consider three circles with radii 2, 3, and 5, respectively. Draw a picture showing these circles so positioned that each is tangent to each of the other two.

(b) Draw pictures showing four other arrangements of the circles which satisfy the condition that each circle is tangent to each of the others.

B. 1.



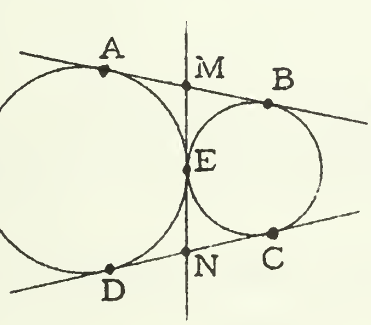
Hypothesis: two circles are tangent externally at T

Conclusion: $T \in \overleftrightarrow{P'P}$

[Hint. By definition, each circle is tangent to $\overleftrightarrow{MN}$ at T.]

2. Repeat Exercise 1 for two circles tangent internally.

3.



Hypothesis: $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$ are common external tangents, the circles are tangent externally at E, $\overleftrightarrow{MN}$ is a common internal tangent

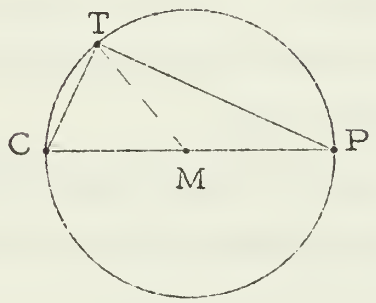
Conclusion: AM = MB

4. In Exercise 3, show that AB = CD. [Consider two cases, one in which the circles have the same radii, and the other in which they have different radii.]

5. Suppose that ℓ and m are two common internal tangents to two circles. If the points of tangency are A, B, C, and D such that A and B belong to ℓ and C and D belong to m, show that AB = CD. [Hint. Assume that $\overline{AB}$ and $\overline{CD}$ intersect at E.]

6. Two circles are externally tangent at T, and ℓ is their common internal tangent. P is a point in ℓ and $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangents to the circles at A and B, respectively. Show that PA = PB.

C. 1.

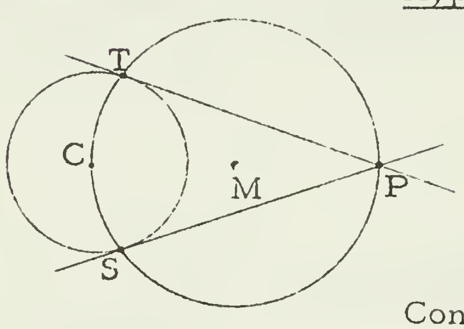


Hypothesis: $\overline{CP}$ is a diameter of a circle with center M,

T is a point of the circle, $T \neq C$, $T \neq P$

Conclusion: $\angle CTP$ is a right angle

2.



Hypothesis: $\overline{CP}$ is longer than a radius of the circle with center C,

$\overline{CP}$ is a diameter of the circle with center M

Conclusion: $\overleftrightarrow{PT}$ and $\overleftrightarrow{PS}$ are tangent to the circle with center C at T and S, respectively

*

A point P is outside a circle with center C and radius r if and only if CP > r. A point outside a circle is sometimes called an external point with respect to the circle.

Give a definition of a point inside a circle [an internal point with respect to the circle].

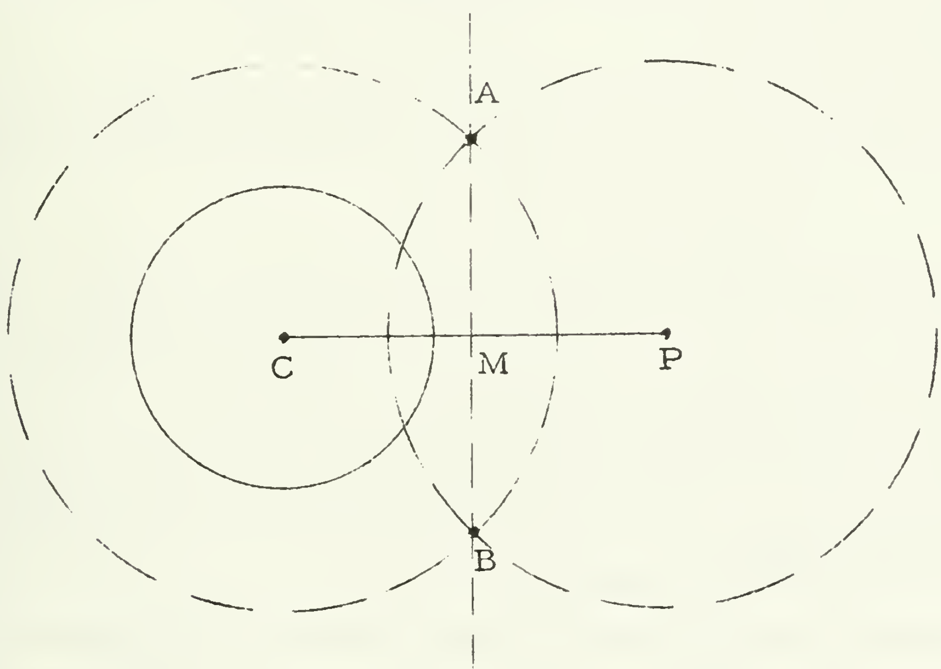
*

3. Draw a circle with center C, and mark a point P outside the circle. Using compass and straight-edge alone, draw a tangent to the circle through P. [Hint. See Exercise 2.]

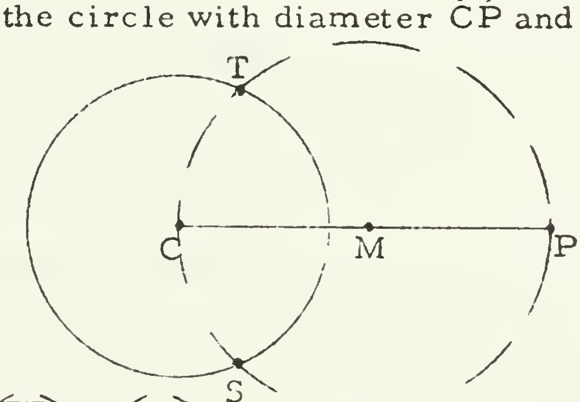
STRAIGHT-EDGE AND COMPASS CONSTRUCTIONS

Throughout this unit you have been asked to draw pictures of geometric figures restricting your drawing tools to a straight-edge and a compass. For example, consider the problem in Exercise 3, that of drawing a line tangent to a circle with center C through an external point P. If you followed the hint given in the exercise, you probably did it this way:

<u>Step (1).</u> Draw $\overline{CP}$ and locate its midpoint.



<u>Step (2).</u> Draw the circle with diameter $\overline{CP}$ and center M.



<u>Step (3).</u> Draw $\overleftrightarrow{PT}$ [or $\overleftrightarrow{PS}$]. Each line is tangent to the circle with center C through P.

That $\overleftrightarrow{PT}$ is a tangent to the circle with center C follows from the conclusion that $\angle CTP$ is a right angle, and this follows from the theorem that a triangle is a right triangle if and only if the measure of a median ($\overline{TM}$) from a vertex is one half the measure of the side ($\overline{CP}$) opposite the vertex.

Now, let's take a careful look at what is involved in solving this problem.

In step (1), we located the midpoint of $\overline{CP}$ by drawing the perpendicular bisector of $\overline{CP}$. We did this by drawing two circles, one with center C and radius greater than half of CP, and the other with center P and the same radius as the first. [Actually, we didn't need to draw the circles completely.] Then, *by looking at the picture*, we see that these circles intersect in two points. Since each of these points is equidistant from C and B, they determine the perpendicular bisector of $\overline{CP}$. [Theorem 3-3]

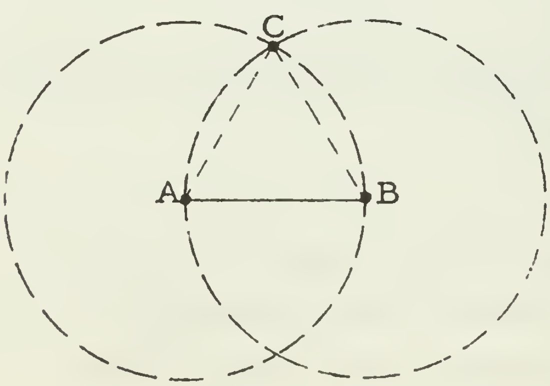
With the help of Theorem 10-12, we can now *prove* that there must be two points of intersection. [Do so.] We do not have to depend on what is merely suggested by the picture.

In step (2), we drew the circle with center M and radius MC. Once, again, this circle intersects the given circle in two points. Not only does the picture suggest it, but we can actually prove that there are such points of intersection. [Do so.]

Finally, since P and T are distinct points [Why?], we know that they determine a line. We can then prove, as we did in Exercise 2 on page 6-292 that $\overleftrightarrow{PT}$ is a tangent.

Problems such as this one which call for locating certain points or lines by using circles and straight lines only [compass and straight-edge] are called *Euclidean construction problems*. These problems occupied a very important place in the geometry of the ancient Greeks.

It is interesting to note that one of the very first construction problems which appeared in Euclid's *Elements* was not correctly solved by Euclid, although he apparently did not realize this. The problem he set was to construct an equilateral triangle given one of its sides. He



proceeded in the obvious manner. Draw the circle with center A and radius AB, and the circle with center B and the same radius. Euclid must then have assumed that the two circles intersect. If C is one of the points of intersection $\triangle ABC$ is equilateral [Why?].

Some historians believe that Euclid was very careful to make explicit all the axioms he felt he needed as a basis for his deductive geometry and to use only these axioms and theorems derived from them in his proofs. If so, this is one of the few places where he failed to live up to his standard.

Can you prove that the two circles with centers A and B and radius AB intersect in two points?

A construction problem has two uses. On the one hand, its solution gives a technique to use in making careful drawings. But, on the other hand it justifies _an existence theorem_. For example, the equilateral triangle construction just described proves that, given any segment, _there are_ exactly two equilateral triangles each of which has this segment as a side.

Example. To construct a line perpendicular to a given line at a given point of the line.

[The construction which forms the solution of this problem shows a simple way of drawing a perpendicular at a point on a line. But, the solution also justifies the theorem:

For each line ℓ and each point $P \in \ell$,
there is a line m such that $P \in m$ and $m \perp \ell$.

Actually, in the way we have organized geometry, the existence of such a line [Theorem 2-8] follows from Axiom E, Theorem 2-1, and the definition of perpendicular lines. The solution to the construction probelm provides another proof.]

Solution. Draw the given line ℓ and mark the given point P on it.
Let A be another point on ℓ. Then, draw the circle with center P and radius PA. This circle intersects ℓ in another point, B. Now, draw two circles with A and B as centers and with a common radius greater than half of PA. These circles intersect in two points. [Prove this.] Let S be one of these points. Then, $\overleftrightarrow{SP}$ is perpendicular to $\overleftrightarrow{AB}$ at P.

To complete the solution, we must justify the last sentence of the preceding paragraph. This is easy to do using the s.s.s. triangle-congruence theorem.

EXERCISES

Here are several straight-edge and compass construction problems. You have already solved most of these in other parts of this unit. So, you should treat these exercises partly as a review. Also, as you solve the problems, keep looking for places where you are actually using the theorems about the intersections of circles and lines.

(1) Construct the figure asked for.

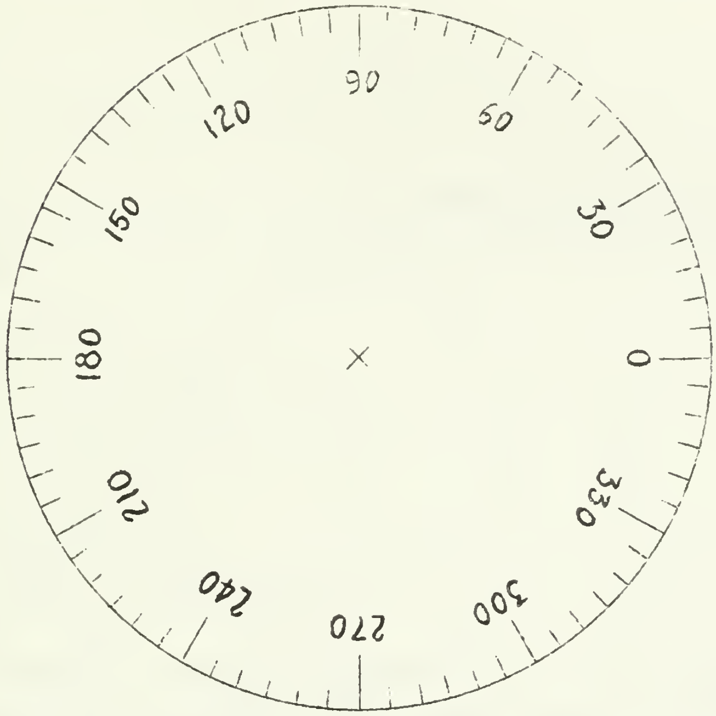
(2) Be ready to prove that the constructed figure is what you claim it to be.

Problems:

1. To bisect a given segment.
2. To bisect a given angle.
3. To construct a line perpendicular to a given line through a given point not on the line.
4. To construct an angle congruent to a given angle and having a given side.
5. To construct a line parallel to a given line through a given point not on the line.
6. To divide a given segment into any given number of congruent segments.
7. To construct a line tangent to a given circle through a given point on the circle.
8. To construct a line tangent to a given circle through a given point outside the circle.
9. To locate the center of a given circle.
10. To construct a circle circumscribed about a given triangle.
11. To construct a circle inscribed in a given triangle.
12. To construct a triangle similar to a given triangle with a given segment as one of the sides of the required triangle.
13. To construct a segment which is a mean proportional between two given segments.

EXPLORATION EXERCISES

Here is a picture of a circular protractor. Imagine that the



protractor is made of transparent plastic. It can be used to measure angles or to help in drawing an angle of a given size.

To measure a pictured angle, $\angle ABC$, place the center of the protractor on B, and note the scale marks where the sides cross the edge. So, for example, if $\overleftrightarrow{BA}$ crosses at the 30-mark and $\overleftrightarrow{BC}$ crosses at the 150-mark, then the degree-measure of $\angle ABC$ is 120.

A. Complete the following table

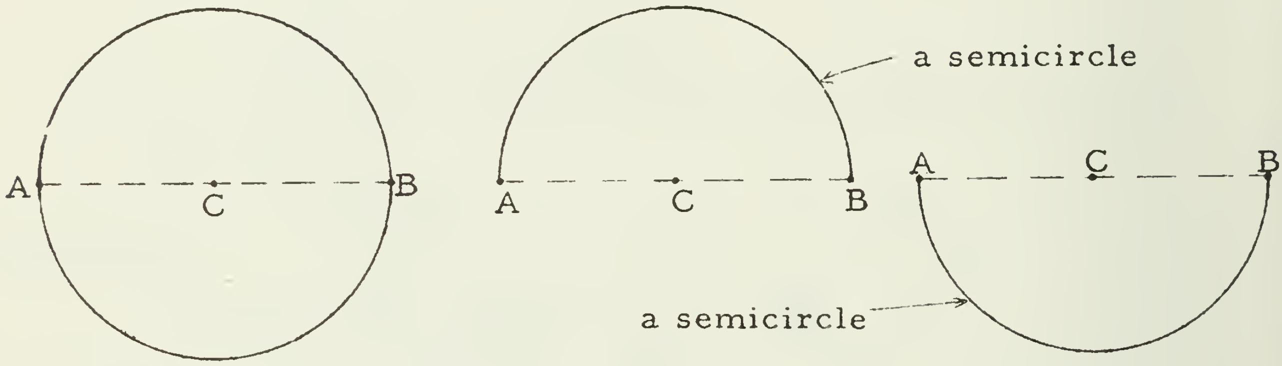
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
$\overleftrightarrow{BA}$	60	60	30	0	310	50	120	180	240
$\overleftrightarrow{BC}$	150	0	300	210	270				
$m(\angle ABC)$						40	130	90	130

B. 1. Suppose that D is a point in the interior of $\angle ABC$. If $\overleftrightarrow{BA}$ crosses at 250 and $\overleftrightarrow{BC}$ crosses at 290, what is $m(\angle ABD) + m(\angle DBC)$?

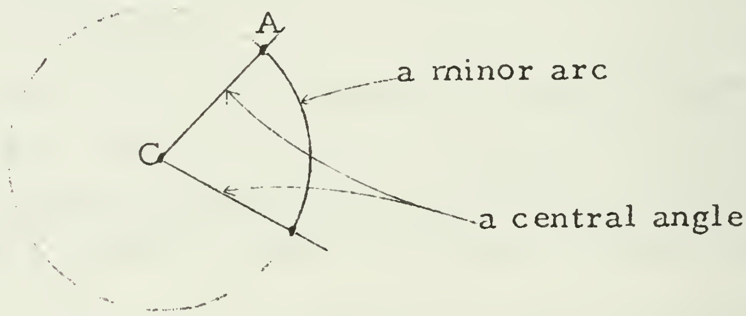
2. Suppose that D is a point in the interior of the vertical angle of $\angle ABC$. If $\overleftrightarrow{BA}$ crosses at 20 and $\overleftrightarrow{BC}$ crosses at 19, what is $m(\angle ABD) + m(\angle DBC)$?

ARCS OF A CIRCLE

Consider a circle with center C. Suppose that A and B are two points of the circle. If A, B, and C are collinear, the set consisting of A and B together with all points of the circle on the same side of line $\overleftrightarrow{AB}$ is called a semicircular arc of the circle, or, for short, a semicircle. A and B are its end points; and C is its center.



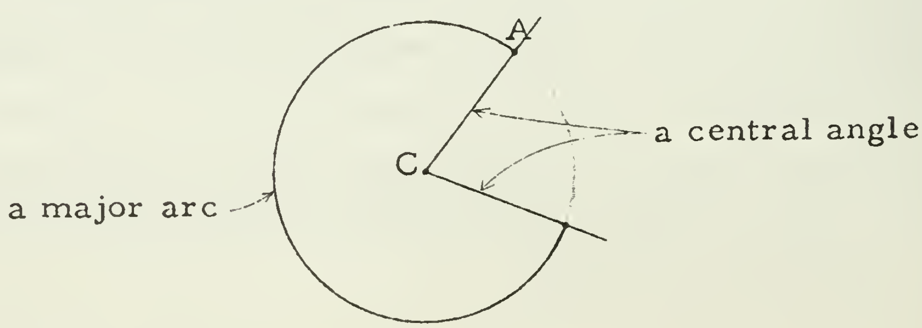
Suppose that A, B, and C are noncollinear. Then consider the rays $\overrightarrow{CA}$ and $\overrightarrow{CB}$. By definition, $\overrightarrow{CA} \cup \overrightarrow{CB}$ is an angle whose vertex is



the center of the circle. It is said to be a central angle of the circle.

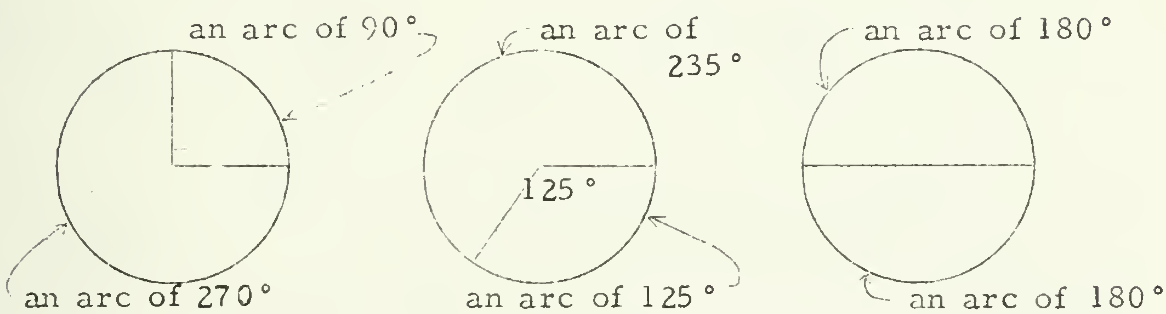
The set consisting of A and B together with all the points of the circle in the interior of the angle is called a minor arc of the circle; A and B are its end points and C is its center.

The set consisting of A and B together with all the points of the circle in the exterior of the angle is called a major arc of the circle; its end points are A and B, and C is its center.



Now, just as we found it convenient to assign measures to certain other geometric figures such as segments and angles, we shall also find it convenient to assign measures to arcs of a circle. We do so according to the following definition:

> For each two points X and Z on a circle with center Y, the measure of the arc with end points X and Z is
>
> $°m(\angle XYZ)$ if the arc is a minor arc,
>
> $360 - °m(\angle XYZ)$ if the arc is a major arc,
>
> and 180 if the arc is a semicircle.



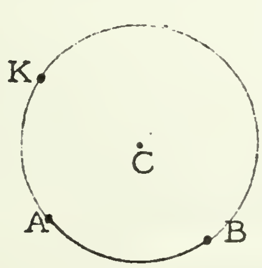
NAMING AN ARC

For each two points A and B of a circle, there are two arcs which have these points as end points. Consequently, unlike the case with segments, if we are to name an arc by just writing the names of its end points like this:

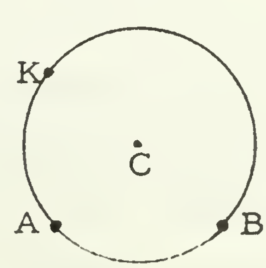
$$\overset{\frown}{AB},$$

we need some convention to tell which of the two arcs we mean.

What we shall do is to use this symbol to name a minor arc. To name a major arc, we shall write the names of the end points and, between them, write a name of another point of the arc.



<u>This is a minor arc, $\overset{\frown}{AB}$.</u>



<u>This is a major arc, $\overset{\frown}{AKB}$.</u>

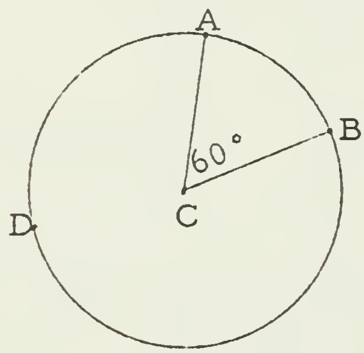
[read '$\overset{\frown}{AB}$' as 'arc AB', and '$\overset{\frown}{AKB}$' as 'arc AKB'.]

Three-point names will also be used for semicircles, and, occasionally, for minor arcs. So, if you see a symbol like '$\overset{\frown}{PQ}$', you know that we are referring to a minor arc. But, if you see a symbol like '$\overset{\frown}{PRQ}$', you can't tell without more information [a figure, for example] whether we're talking about a minor arc, a major arc, or a semicircle.

EXERCISES

A. Find the indicated measures.

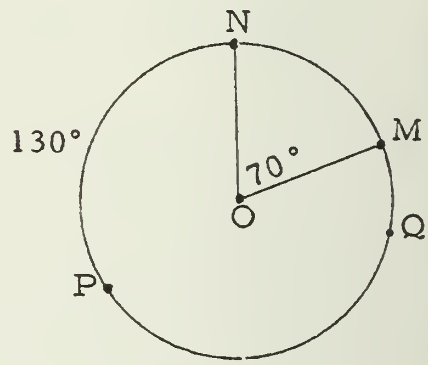
1.



(a) $m(\overset{\frown}{AB})$ = _____

(b) $m(\overset{\frown}{ADB})$ = _____

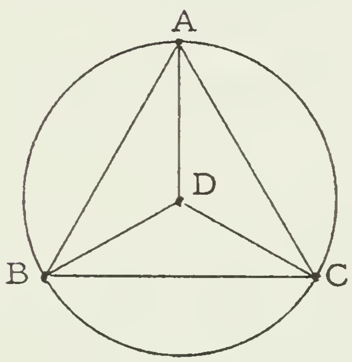
2.



(a) $m(\angle NOP)$ = _____

(b) $m(\overset{\frown}{MQP})$ = _____

3.



Given: $\triangle ABC$ is equilateral, D is the center of its circumcircle

Find: (a) $m(\overset{\frown}{AB})$ = _____

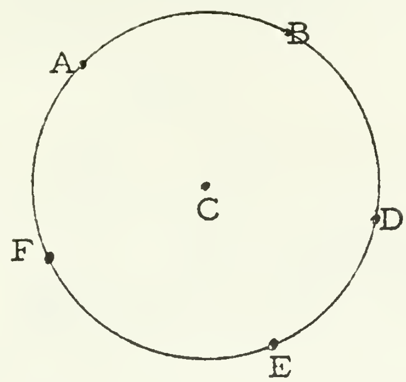
(b) $m(\overset{\frown}{ABC})$ = _____

B. Suppose that the points A, B, D, E, and F are points on a circle with center C, and ordered as in the diagram on page 6-301.

1. If $m(\overset{\frown}{ABD}) = 110$, and $m(\overset{\frown}{EA}) = 100$, what is $m(\overset{\frown}{ED})$?

2. If $m(\overset{\frown}{AB}) = m(\overset{\frown}{BD}) = m(\overset{\frown}{DE}) = m(\overset{\frown}{EF}) = m(\overset{\frown}{FA})$, what is $m(\overset{\frown}{AB})$?

3. If $m(\overset{\frown}{FAB}) = 2 \cdot m(\overset{\frown}{BDF})$, what is $m(\overset{\frown}{BAF})$?



4. What arc is $\overset{\frown}{FAB} \cap \overset{\frown}{ABD}$?

5. What arc is $\overset{\frown}{FAB} \cup \overset{\frown}{ABD}$?

6. What is $\overset{\frown}{FAB} \cap \overset{\frown}{BDE}$?

7. What is $\overset{\frown}{FAB} \cup \overset{\frown}{BDF}$?

8. What is $(\overset{\frown}{FAB} \cup \overset{\frown}{BDE}) \cup \overset{\frown}{AFE}$?

9. If $\overset{\frown}{ABD}$ is an arc of 170° and $\overset{\frown}{FEA}$ is an arc of 280°, what is $m(\overset{\frown}{AE})$?

10. If $\overset{\frown}{ABD}$ is an arc of 160° and $\overset{\frown}{BDE}$ is an arc of 160°, what is $m(\overset{\frown}{AE})$?

11. If $\overset{\frown}{ABD}$ and $\overset{\frown}{EDB}$ are semicircles, and $m(\overset{\frown}{AB}) = m(\overset{\frown}{BD})$, what are $m(\overset{\frown}{EFA})$ and $m(\overset{\frown}{ABD} \cap \overset{\frown}{BDE})$?

12. If $m(\angle ACB) = 40$ and $m(\angle BCE) = 120$, what are $m(\overset{\frown}{EFA})$ and $m(\angle ECA)$?

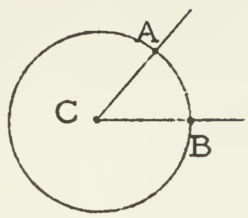
13. If $m(\overset{\frown}{AED}) = 210$ and $m(\overset{\frown}{BD}) = 70$, what are $m(\angle ACB)$ and $m(\overset{\frown}{ACD})$?

* * *

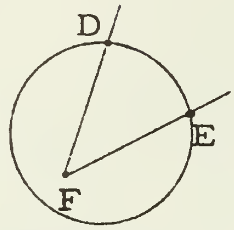
Two circles are congruent if and only if they have the same radius. Also, a circle is congruent to itself.

Two arcs are congruent if and only if they are arcs of congruent circles and have the same measure. An arc is congruent to itself.

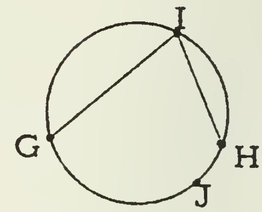
An arc is said to be <u>intercepted by a given angle</u> if and only if its end points belong to the angle and its other points are interior to the angle.



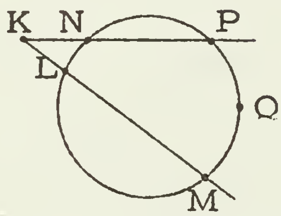
$\angle C$ intercepts $\overset{\frown}{AB}$



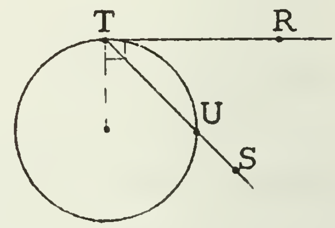
$\angle F$ intercepts $\overset{\frown}{DE}$



$\angle I$ intercepts $\overset{\frown}{HJG}$



$\angle K$ intercepts $\overset{\frown}{LN}$ and $\overset{\frown}{MQP}$



$\angle RTS$ intercepts $\overset{\frown}{TU}$

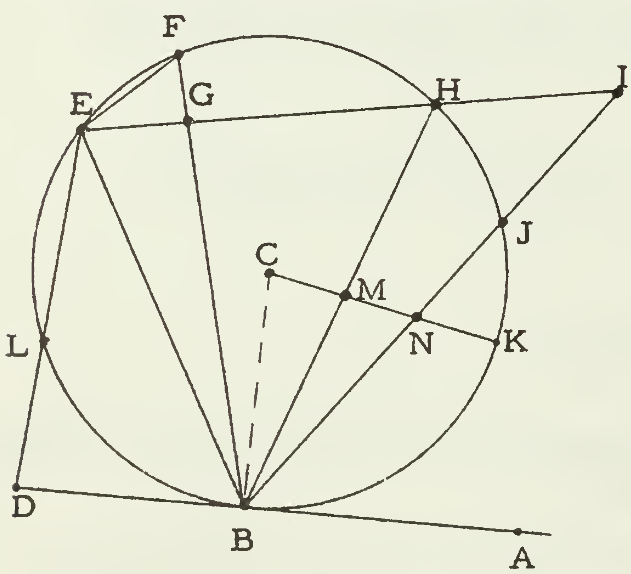
An angle is said to be <u>inscribed in a given arc</u> if and only if its sides contain the end points of the arc and its vertex is another point of the arc. In the figures shown above, $\angle I$ is inscribed in $\overset{\frown}{GIH}$. Also, $\angle MLP$ is inscribed in $\overset{\frown}{MNP}$ and so is $\angle MNP$. [Tell why $\angle RTS$ is not inscribed in $\overset{\frown}{TU}$.]

An angle is said to be <u>an inscribed angle of a given circle</u> if and only if it is inscribed in an arc of the circle.

* * *

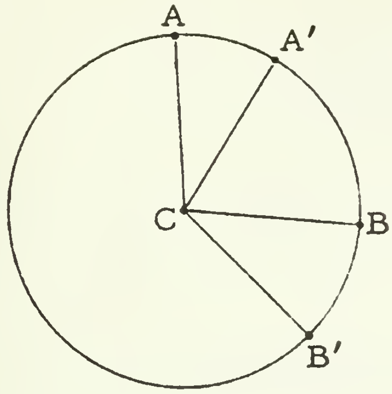
<u>C.</u> Complete the sentences.

<u>Given:</u> C is the center, $\overleftrightarrow{CB} \perp \overleftrightarrow{AB}$



1. $\angle EFG$ intercepts ______
2. $\angle EFG$ is inscribed in ______
3. $\angle I$ intercepts ______
4. $\angle C$ intercepts ______
5. $\overset{\frown}{ELB}$ is intercepted by ______
6. $\angle ABI$ intercepts ______
7. $\angle ABE$ intercepts ______
8. $\angle HGB$ intercepts ______
9. $\angle FGE$ intercepts ______
10. $\angle EBJ$ is inscribed in ______
11. $\angle FBH$ is inscribed in ______
12. $\angle D$ intercepts ______

Example.



Hypothesis: C is the center of the circle, $\angle ACB \cong \angle A'CB'$

Conclusion: $\overset{\frown}{AB} \cong \overset{\frown}{A'B'}$, $\overset{\frown}{AB'B} \cong \overset{\frown}{A'AB'}$

Solution I.

(1)	$\angle ACB$ and $\angle A'CB'$ are central angles	[Hypothesis; def. of central angle]
(2)	$\angle ACB \cong \angle A'CB'$	[Hypothesis]
(3)	$m(\angle ACB) = m(\angle A'CB')$	[(2); def. of congruent angles]
(4)	$m(\angle ACB) = m(\overset{\frown}{AB})$	[(1); def. of arc-measure]
(5)	$m(\angle A'CB') = m(\overset{\frown}{A'B'})$	[(1); def. of arc-measure]
(6)	$m(\overset{\frown}{AB}) = m(\overset{\frown}{A'B'})$	[(3), (4), and (5)]
(7)	$\overset{\frown}{AB} \cong \overset{\frown}{A'B'}$	[(6); def. of congruent arcs]
(8)	$m(\overset{\frown}{AB'B}) = 360 - m(\angle ACB)$	[(1); def. of arc-measure]
(9)	$m(\overset{\frown}{A'AB}) = 360 - m(\angle A'CB')$	[(1); def. of arc-measure]
(10)	$m(\overset{\frown}{AB'B}) = m(\overset{\frown}{A'AB'})$	[(8), (9), and (3)]
(11)	$\overset{\frown}{AB'B} \cong \overset{\frown}{A'AB'}$	[(10); def. of arc-measure]

Solution II.

Since, by hypothesis, $\angle ACB$ and $\angle A'CB'$ are congruent central angles, they have the same measure. So, the arcs $\overset{\frown}{AB}$ and $\overset{\frown}{A'B'}$ which they intercept have the same measure. Hence, these arcs are congruent. For the same reason, the corresponding major arcs are congruent.

A similar argument establishes the following theorem:

Theorem 10-15.

Arcs of the same or congruent circles which are intercepted by congruent central angles are congruent.

Other theorems which are easily derived from the definitions of central angle, arc-measure, angle-congruence, and arc-congruence are:

> Theorem 10-16.
>
> Semicircles of the same or congruent circles are congruent.

> Theorem 10-17.
>
> Major arcs of the same or congruent circles are congruent if and only if the related minor arcs are congruent.

> Theorem 10-18.
>
> Central angles of the same or congruent circles which intercept congruent arcs are congruent.

[Contrast Theorems 10-15 and 10-18.]

* * *

D. 1. Prove:

> Theorem 10-19.
>
> Minor arcs of the same or congruent circles are congruent if and only if their chords are congruent.

[The chord of an arc of a circle is the chord of the circle whose end points are the end points of the arc.]

2. $\overline{AB}$ and $\overline{CD}$ are chords of a circle with center P. If $\overline{AB}$ is longer than $\overline{CD}$, show that $m(\overset{\frown}{AB}) > m(\overset{\frown}{CD})$ and that $\overline{AB}$ is closer to P than $\overline{CD}$.

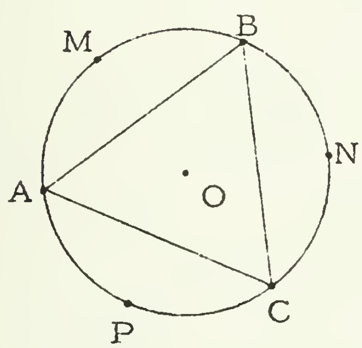
3. If $\overline{AB}$ and $\overline{CD}$ are chords of a circle with center P and $\overline{AB}$ is closer to P than $\overline{CD}$, show that $\overline{AB}$ is longer than $\overline{CD}$.

E. Prove:

> Theorem 10-20.
>
> A diameter perpendicular to a chord of the circle bisects both arcs whose end points are those of the chord.

[A set s is said to bisect an arc with end points A and B if and only if s intersects the arc at a single point P such that $\overset{\frown}{AP} \cong \overset{\frown}{PB}$. The point P is called the midpoint of the arc.]

F. 1.



Hypothesis: $\triangle ABC$ is equilateral, and a circle with center O is its circumcircle,
M, N, and P are the midpoints of $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, and $\overset{\frown}{CA}$, respectively

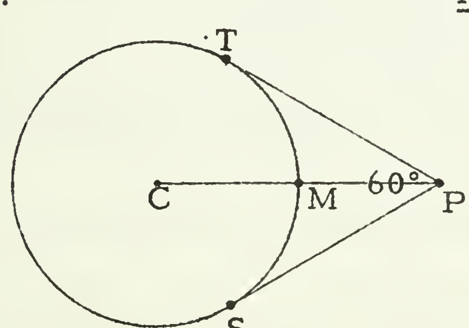
Conclusion: AMBNCP is a regular hexagon

2. Suppose that $\overline{AB}$ is a diameter of a circle with center O, and $\overline{CD}$ is a chord of this circle and perpendicular to $\overline{AB}$. If $\angle ACD$ is an angle of 80°, how many degrees are there in $\angle DAB$?

3. In Exercise 2, if $\angle ACD$ is an angle of 60°, show that $\overset{\frown}{AC} \cong \overset{\frown}{CD} \cong \overset{\frown}{DA}$.

4. Suppose that $\overline{AB}$ and $\overline{BC}$ are two congruent chords of a circle with center O. Show that $\overrightarrow{BO}$ is the angle bisector of $\angle ABC$.

5.



Hypothesis: $\overleftrightarrow{PT}$ and $\overleftrightarrow{PS}$ are tangents to a circle with center C,
$m(\angle TPS) = 60$

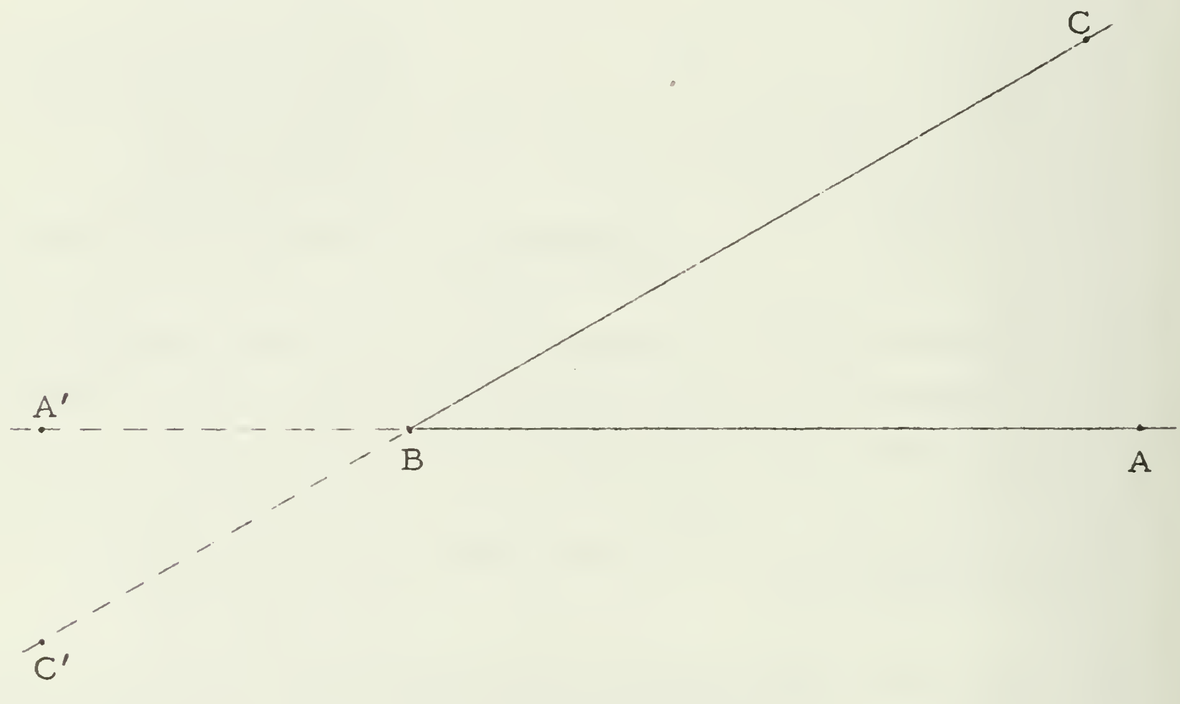
Conclusion: $CM = MP$

EXPLORATION EXERCISES

Turn to page 6-297 and look at the picture of the circular protractor. Make your own circular protractor [1.5'' in radius] by cutting a circular disc out of heavy paper and roughly calibrating it by using what you know about arc-measure.

Now, this circular protractor differs from the one described on page 6-297 by not being transparent. So, you can't use it to measure an angle in the way described in the text. However, it can still be used in several ways to measure angles.

<u>A.</u> Here is a picture of $\angle ABC$. [You could measure this angle by using a regular protractor or a transparent circular protractor, but don't do it, yet.]



1. Place the protractor on the picture in such a way that the 180-mark is on B and $\overleftrightarrow{BA}$ crosses the edge at:

$\overleftrightarrow{BA}$	0	10	30	45	80	90	100	110	120
$\overleftrightarrow{BC}$									

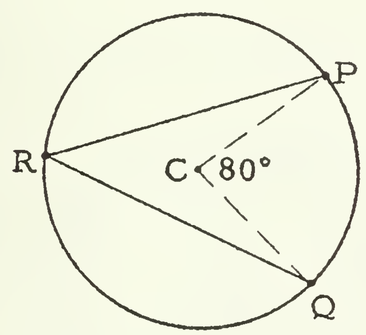
Record in the bottom half of this table the scale marks where $\overleftrightarrow{BC}$ crosses the edge.

2. Study the table for a clue to the size of $\angle ABC$. What do you think the measure of $\angle ABC$ is? Check with a regular protractor.

3. Now, place the protractor so that B is at some point on the edge other than the 180-mark, and both $\overrightarrow{BA}$ and $\overrightarrow{BC}$ cross the edge. Record the scale marks for $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Do this several times. Do you see a way to use the scale marks to compute the measure of $\angle ABC$? Test your discovery on some other angles including a right angle and an obtuse angle.

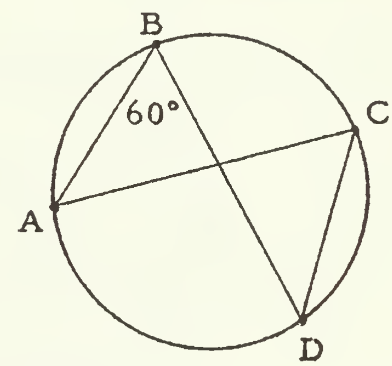
4. Find the indicated measures.

(a)



$m(\overset{\frown}{PQ})$ = _____,
$m(\angle R)$ = _____

(b)



$m(\overset{\frown}{AD})$ = _____,
$m(\angle C)$ = _____

<u>B.</u> Return, now, to the given angle, $\angle ABC$. Place your protractor so that the 180-mark is on B and $\overrightarrow{BA}$ crosses the edge at the 0-mark. As you discovered earlier, $\overrightarrow{BC}$ crosses the edge at the 60-mark. Now, shift your protractor about 3/4 of an inch to the left, keeping the 0- and 180-marks on the line $\overleftrightarrow{AB}$.

1. What are the scale marks for $\overrightarrow{BC}$ and $\overrightarrow{BC'}$?

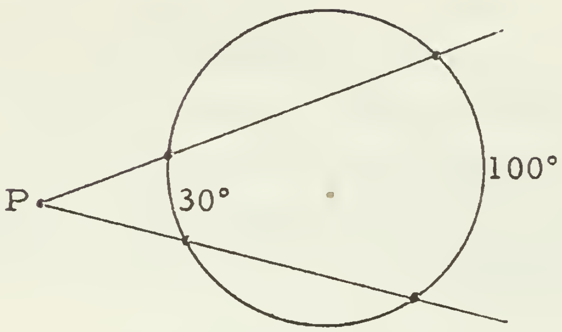
2. Can you use these scale marks to compute $m(\angle ABC)$?

3. Shift the protractor 1/8 of an inch further to the left, and now use the new scale marks for $\overrightarrow{BC}$ and $\overrightarrow{BC'}$ to compute the measure of $\angle ABC$.

<u>C.</u> Repeat Part B but move the protractor to the right instead of to the left. Test the discoveries you made in Parts B and C on several other angles without keeping the 0- and 180-marks on line $\overleftrightarrow{AB}$.

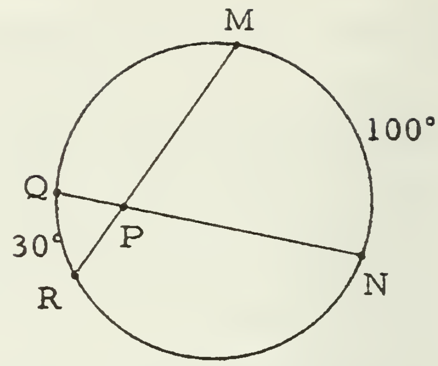
D. Find the indicated measures.

1.



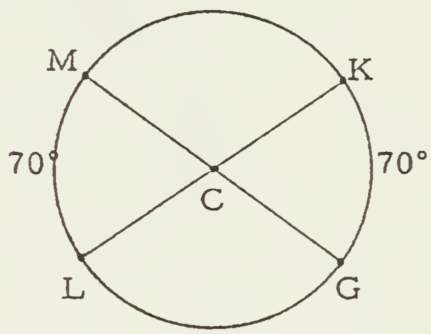
$m(\angle P) =$ _______

2.

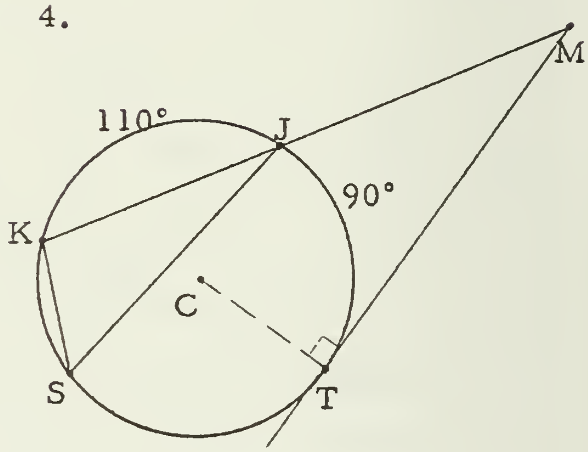


$m(\angle MPN) =$ _______

3.



4.



$m(\angle KGC) =$ _____ ,

$m(\angle GCL) =$ _____

$m(\angle S) =$ _____ ,

$m(\overset{\frown}{KST}) =$ _____ , $m(\angle M) =$ _____

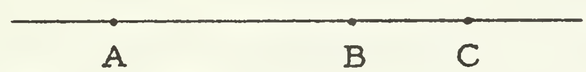
E. In the preceding parts, you learned a variety of ways to use a circular protractor to measure an angle. It turns out that if the vertical angle of the given angle is drawn, you can measure the angle by placing the protractor in any position as long as both of the lines containing the sides cross the edge. In case you haven't already done so, try placing the protractor so that both lines are tangent to the edge. Can you use the scale marks at the points of tangency to compute the measure?

F. Suppose someone covers the vertex of the angle with the circular protractor so that all you can see is that the sides cross the edge at the 0- and 180-marks. What can you conclude about the measure of the angle?

MEASURES OF ANGLES and MEASURES OF THEIR INTERCEPTED ARCS

The results you discovered in the Exploration Exercises can be stated in terms of measures of angles and measures of the arcs they intercept on a circle. In order to prove such theorems we need another theorem about arc-measures.

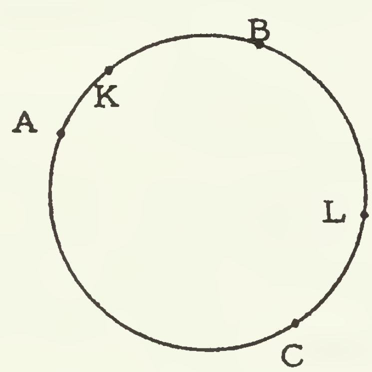
The theorem we need is analogous to Axiom A. Recall that Axiom A tells us that given two segments on a line, if they have only an end



point in common, then the sum of their measures is the measure of their union:

$$m(\overline{AB}) + m(\overline{BC}) = m(\overline{AC})$$

The theorem about arcs says that given two arcs on a circle, if



they have only an end point in common, then the sum of their measures is the measure of their union:

$$m(\overset{\frown}{AKB}) + m(\overset{\frown}{BLC}) = m(\overset{\frown}{ABC})$$

$$m(\overset{\frown}{KAL}) + m(\overset{\frown}{LB}) = \underline{\qquad\qquad}$$

The proof is fairly complicated, and we shall not take time to give it. Your knowledge of the sum of the measures of adjacent angles [see page 6-70] would take care of the case in which $\overset{\frown}{AB}$ and $\overset{\frown}{BLC}$ are minor arcs. But, you would need to use some Introduction Theorems to take care of the case in which $\overset{\frown}{AKB}$ is a minor arc and $\overset{\frown}{BLC}$ is a major arc.

We shall accept this theorem without proof.

> Theorem 10-21.
>
> The sum of the measures of two concyclic arcs which have only an end point in common is the measure of their union.

[A set of concyclic points is a set which is a subset of some circle. We often speak of two sets as being concyclic if their union is a set of concyclic points. So, for example, two arcs are concyclic if and only if they are arcs of the same circle.]

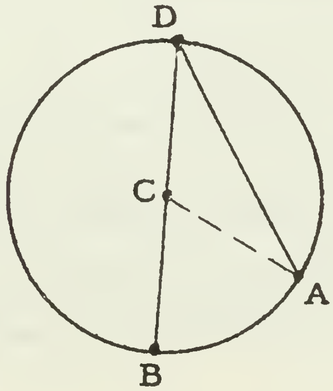
Probably the first of the theorems you discovered in the Exploration Exercises is the following:

> Theorem 10-22.
>
> The measure of an inscribed angle is half the measure of its intercepted arc.

You will have established this theorem and some related ones in the next set of exercises. In doing these exercises, you will find it helpful to use the results of some of them in proving others. In fact, you may even want to rearrange the exercises to take more advantage of this possibility.

EXERCISES

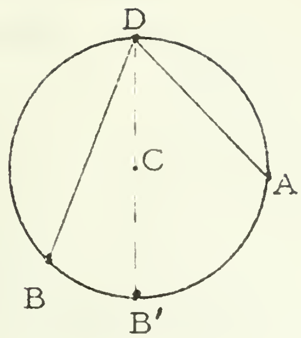
A. 1.



Hypothesis: $\angle ADB$ is inscribed in a circle with center C, $C \in \overline{BD}$

Conclusion: $m(\angle ADB) = \frac{1}{2} \cdot m(\widehat{AB})$

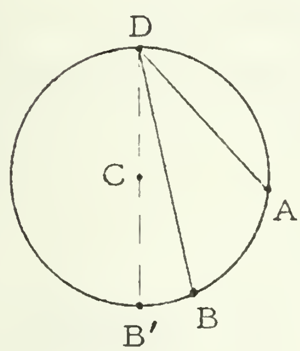
2.



Hypothesis: $\angle ADB$ is inscribed in a circle with center C, A and B are on opposite sides of $\overleftrightarrow{DC}$

Conclusion: $m(\angle ADB) = \frac{1}{2} \cdot m(\overset{\frown}{AB'B})$

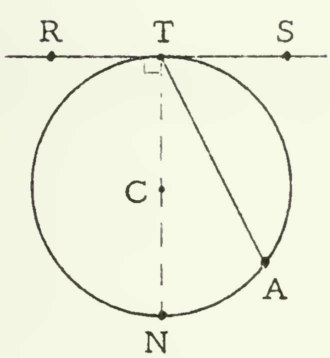
3.



Hypothesis: $\angle ADB$ is inscribed in a circle with center C, A and B are on the same side of $\overleftrightarrow{DC}$

Conclusion: $m(\angle ADB) = \frac{1}{2} \cdot m(\overset{\frown}{AB})$

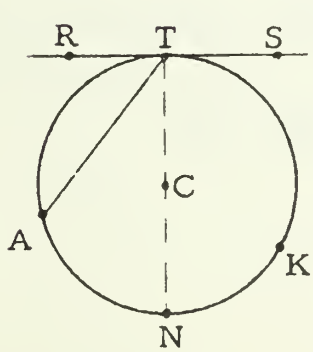
4.



Hypothesis: $\overleftrightarrow{RS}$ is a tangent at T to a circle with center C

Conclusion: $m(\angle STA) = \frac{1}{2} \cdot m(\overset{\frown}{TA})$

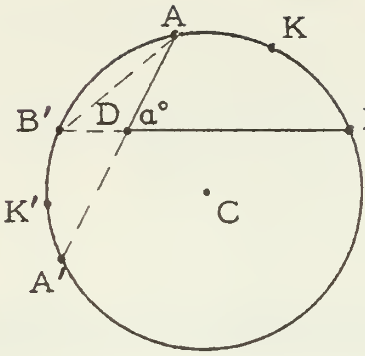
5.



Hypothesis: $\overleftrightarrow{RS}$ is tangent at T to a circle with center C

Conclusion: $m(\angle STA) = \frac{1}{2} \cdot m(\overset{\frown}{TKA})$

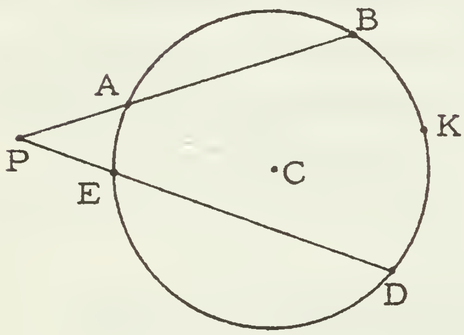
6.



Hypothesis: D is an internal point of a circle with center C,
$D \in \overline{B'B}$,
$D \in \overline{A'A}$,
$m(\angle ADB) = a$

Conclusion: $a = \dfrac{m(\overset{\frown}{AKB}) + m(\overset{\frown}{A'K'B'})}{2}$

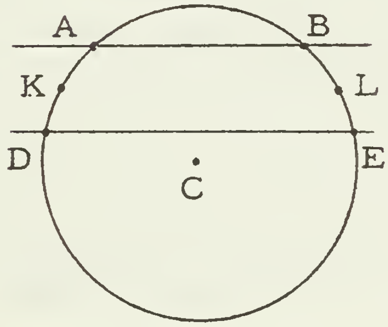
7.



Hypothesis: $\overleftrightarrow{PB}$ and $\overleftrightarrow{PD}$ are secants of a circle with center C

Conclusion: $m(\angle P) = \dfrac{m(\overset{\frown}{BKD}) - m(\overset{\frown}{AE})}{2}$

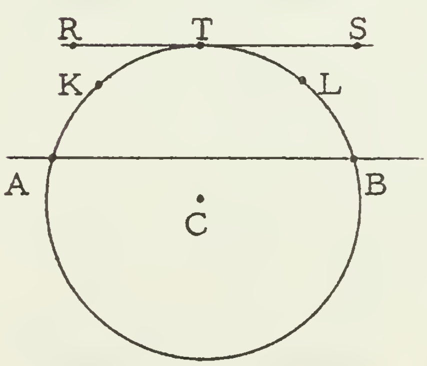
8.



Hypothesis: C is the center of a circle,
$\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$

Conclusion: $\overset{\frown}{AKD} \cong \overset{\frown}{BLE}$

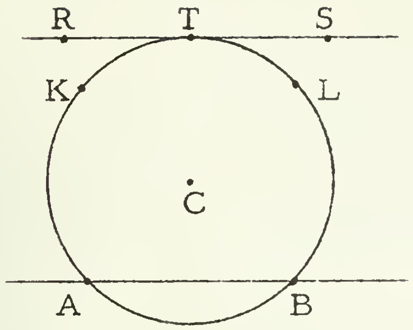
9.



Hypothesis: $\overleftrightarrow{RS}$ is tangent at T to a circle with center C,
$\overleftrightarrow{RS} \parallel \overleftrightarrow{AB}$

Conclusion: $\overset{\frown}{AKT} \cong \overset{\frown}{TLB}$

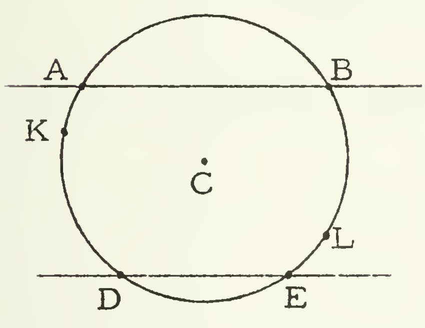
10.



<u>Hypothesis</u>: [as in Exercise 9]

<u>Conclusion</u>: [as in Exercise 9]

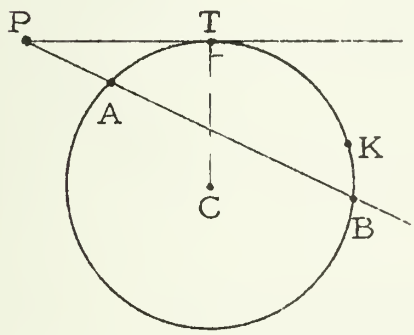
11.



<u>Hypothesis</u>: [as in Exercise 8]

<u>Conclusion</u>: [as in Exercise 8]

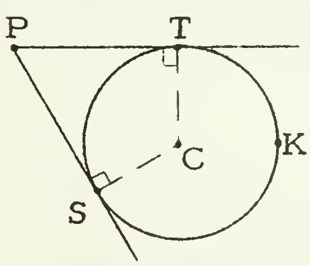
12.



<u>Hypothesis</u>: $\overleftrightarrow{PT}$ is tangent at T to a circle with center C, $\overleftrightarrow{PB}$ is a secant

<u>Conclusion</u>: $m(\angle P) = \dfrac{m(\overset{\frown}{TKB}) - m(\overset{\frown}{TA})}{2}$

13.



<u>Hypothesis</u>: $\overleftrightarrow{PT}$ and $\overleftrightarrow{PS}$ are tangents at T and S, respectively, to a circle with center C

<u>Conclusion</u>: $m(\angle P) = \dfrac{m(\overset{\frown}{TKS}) - m(\overset{\frown}{TS})}{2}$

$= 180 - m(\overset{\frown}{TS})$

* * *

<u>Theorem 10-23.</u> [Tangent-chord theorem]

The measure of an angle one of whose sides is tangent to a circle at the vertex and whose other side intersects the circle is half the measure of the intercepted arc.

<u>Theorem 10-24.</u> [Two-chord theorem]

The measure of an angle whose vertex is an internal point of a circle is half the sum of the measures of the arcs intercepted by the angle and its vertical angle.

<u>Theorem 10-25.</u> [Two-secant theorem]

The measure of an angle whose vertex is an external point of a circle and whose sides intersect the circle is half the difference of the intercepted arcs.

<u>Theorem 10-26.</u> [Two-tangent theorem]

An angle whose sides are tangent to a circle is a supplement of the central angle whose sides contain the points of tangency.

<u>Theorem 10-27.</u>

Parallel lines which intersect a circle cut off congruent arcs.

* * *

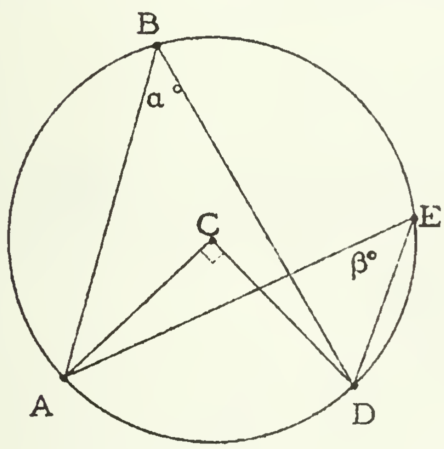
B. Prove each of the following theorems.

1. **Theorem 10-28.** Angles inscribed in the same arc or congruent arcs are congruent.

2. **Theorem 10-29.** An angle inscribed in a semicircle is a right angle.

C. Find the indicated measures.

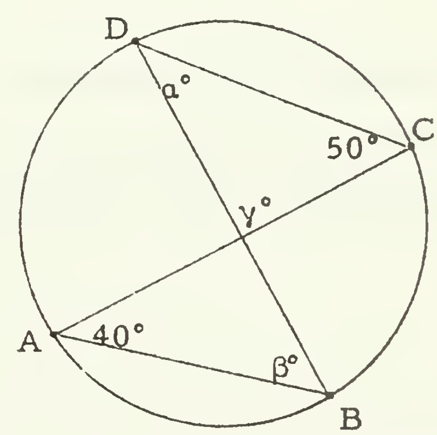
1.



$m(\overset{\frown}{AD})$ = ______ ,

a = ______ , β = ______

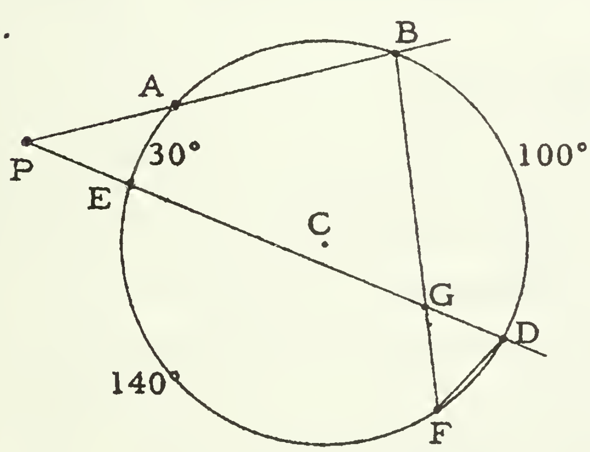
2.



a = ______ , β = ______ ,

$m(\overset{\frown}{AB}) + m(\overset{\frown}{DC})$ = ______

3.

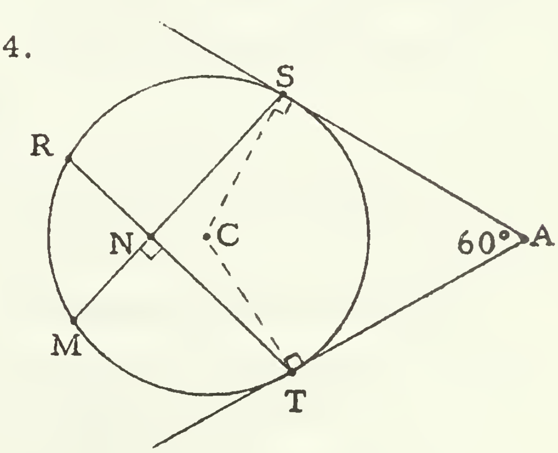


$m(\angle P)$ = ______ ,

$m(\angle BGP)$ = ______ ,

$m(\overset{\frown}{AB})$ = ______

4.

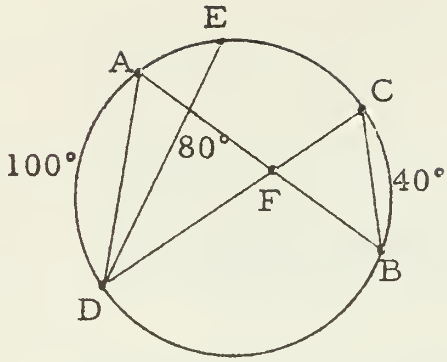


$m(\overset{\frown}{ST})$ = ______ ,

$m(\overset{\frown}{SRT})$ = ______ ,

$m(\angle ASN)$ = ______

5.

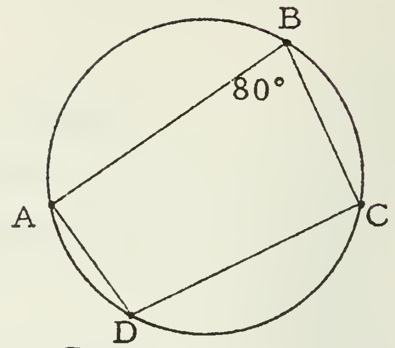


$m(\angle CFB) =$ _____,

$m(\overset{\frown}{EC}) =$ _____,

$m(\angle DAB) + m(\angle CBA) =$ _____

6.



$m(\overset{\frown}{ADC}) =$ _____,

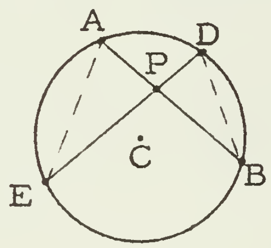
$m(\overset{\frown}{ABC}) =$ _____,

$m(\angle D) =$ _____

[Supplementary exercises are on page 6-438.]

SEGMENTS AND CIRCLES

Theorem 10-28 has an interesting consequence. Suppose P is an internal point with respect to a circle with center C, and $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$



are any two chords which contain P. With the help of Theorem 10-28, you can easily show that $AEP \longleftrightarrow DBP$ is a similarity. Do so.

From this it follows that

$$\frac{AP}{DP} = \frac{PE}{PB}.$$

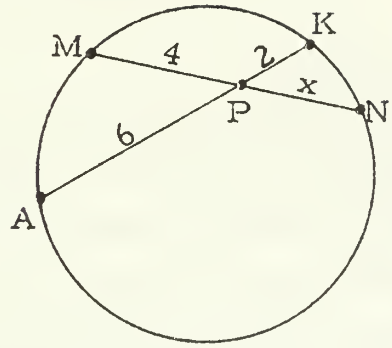
So, by algebra, we have that

$$AP \cdot PB = DP \cdot PE.$$

Thus, we have the following theorem:

> <u>Theorem 10-30.</u>
>
> If two chords of a circle intersect, the point of intersection divides each chord into segments such that the product of the measures of the segments of one chord is the product of the measures of the segments of the other.

Example 1. Find PN.



Solution. By Theorem 10-30,

$$4x = 6 \cdot 2.$$

So, PN = 3.

* * *

Theorem 10-30 can be interpreted in another way. Suppose P is an internal point with respect to the circle with center C. Consider all the lines which contain P [see Figure 1]. Each of these lines intersects the circle in two points such that the product of the distances between P and the points of intersection is the same for all lines.

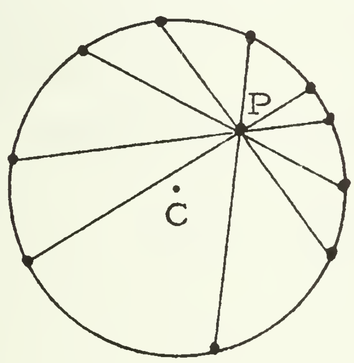


Figure 1

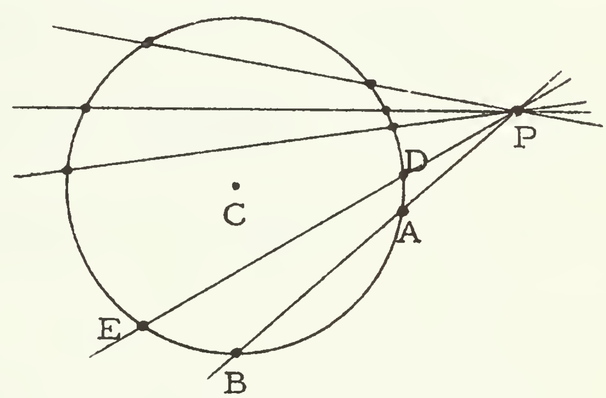


Figure 2

Now, let's look at Figure 2 in which P is an external point. Consider the lines through P, each of which intersects the circle in two points. One such line is the secant $\overleftrightarrow{PB}$. Another is the secant $\overleftrightarrow{PE}$.

For the secant $\overleftrightarrow{PB}$, the distances between P and the points of intersection, A and B, are PA and PB, respectively.

For the secant $\overleftrightarrow{PE}$, the distances are PD and PE, respectively.

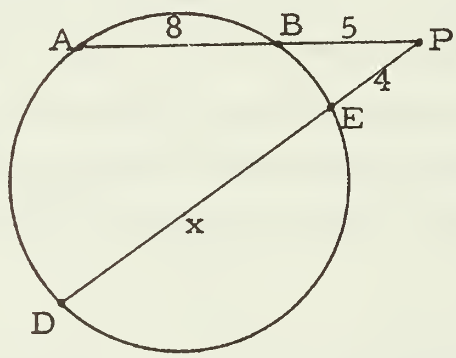
Is it the case that $PA \cdot PB = PD \cdot PE$? The answer to this question is 'yes', and the proof proceeds along lines similar to the case in which P is an internal point. Carry out the proof.

So, we now have the following theorem:

> Theorem 10-31.
>
> If two secants of a circle intersect in an external point, the product of the distances between the external point and the points of intersection on one secant is the product of the distances between the external point and the points of intersection on the other secant.

* * *

Example 2. Find DE.



Solution. It follows from Theorem 10-31 that

$$PB \cdot PA = PE \cdot PD.$$

So,

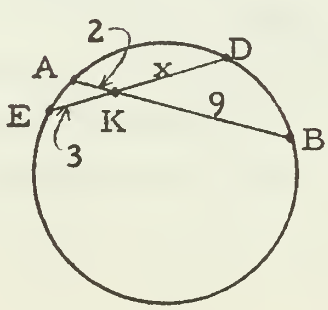
$$5(5 + 8) = 4(4 + x)$$
$$65 = 16 + 4x$$
$$49 = 4x$$
$$12.25 = x.$$

So, DE = 12.25.

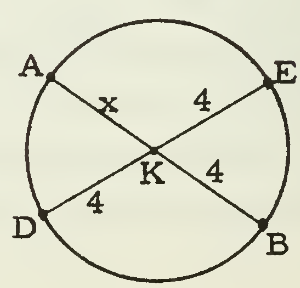
EXERCISES

A. Compute the missing measures.

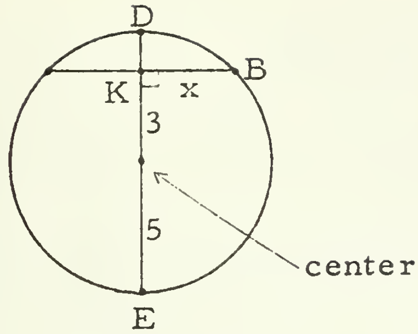
1.



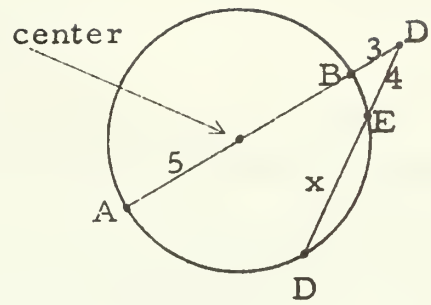
2.



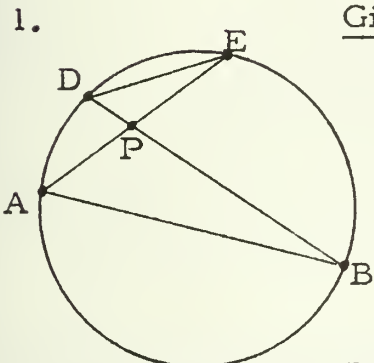
3.



4.



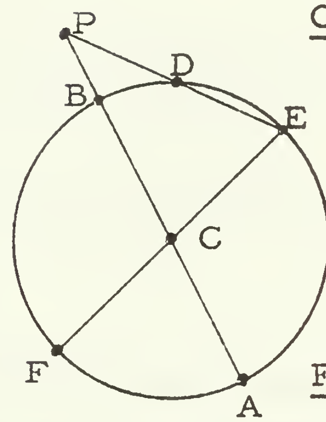
B. 1.



Given:
AE = 7,
AP = 4,
PB = 8,
AB = 10

Find: DE

2.

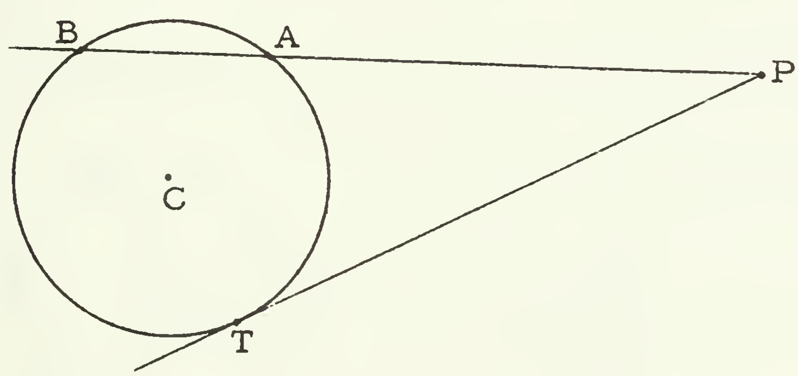


Given:
PD = 5,
PB = 3,
DE = 7,
C is center

Find: EF

* * *

Let's look again at the case in which P is an external point. Suppose two lines, one a tangent to a circle with center C, and the



other a secant of the circle, intersect at P. For the tangent $\overleftrightarrow{PT}$, the "distances" between P and the circle are PT and PT. For the secant $\overleftrightarrow{PB}$, the distances between P and the circle are PA and PB. Show that $PT \cdot PT = PA \cdot PB$. [Hint. Consider $\triangle ATP$ and $\triangle TBP$.]

The segment $\overline{PT}$ is called a tangent segment. The segment $\overline{PB}$ is called a secant segment, and the segment $\overline{PA}$ is called the external segment of secant $\overleftrightarrow{PB}$.

Since, as you have shown above,

$$(PT)^2 = PA \cdot PB,$$

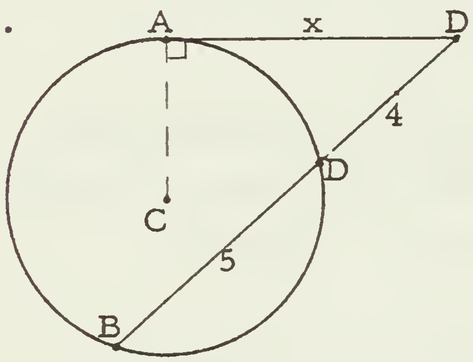
$\overleftrightarrow{PT}$ is the mean proportional between $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$. So, we have the following theorem:

> Theorem 10-32.
>
> If a secant and a tangent of a circle intersect in an external point, then the tangent segment is the mean proportional between the secant segment and the external segment of the secant.

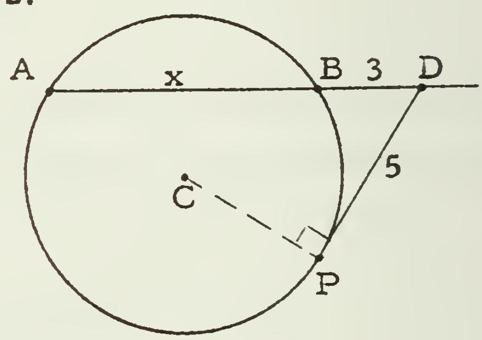
EXERCISES

A. Compute the missing measures. [Assume that C is the center.]

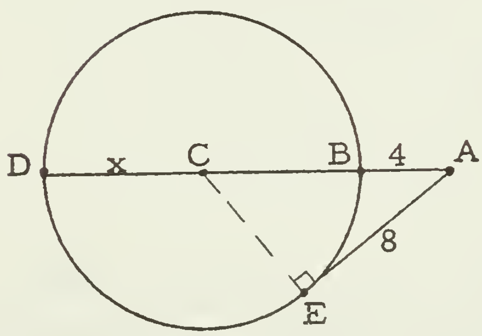
1.



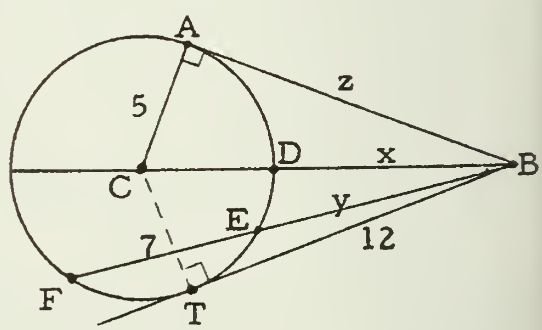
2.



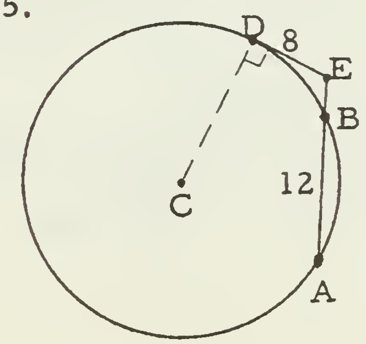
3.



4.

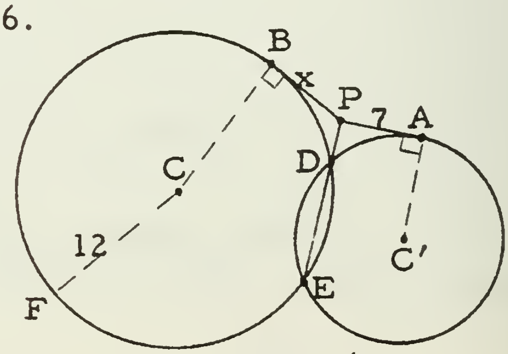


5.



EB = ?

6.



DE = 6

B. 1. In a 14-inch circle [radius 14], a chord 20 inches long is drawn through a point 10 inches from the center. How long is each of the segments of the chord?

2. Two tangents are drawn from an external point P to a 3-inch circle. Let A and B be the points of tangency. If one of the tangent segments is 4 inches long, how long is the altitude of $\triangle PAB$ from P?

3. Prove: Two secant segments from a common external point are congruent if and only if their chords are congruent.

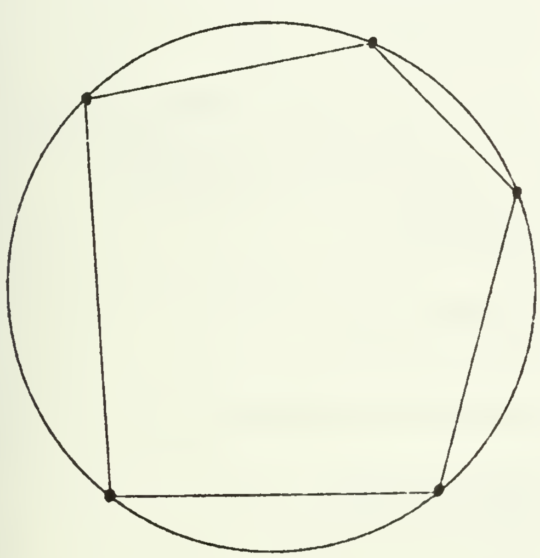
4. Prove: A tangent segment from a point P is longer than the external segment of any secant from that point.

INSCRIBED AND CIRCUMSCRIBED POLYGONS

If the vertices of a polygon belong to a circle then the polygon is said to be inscribed in the circle, and the circle is said to be circumscribed about the polygon.

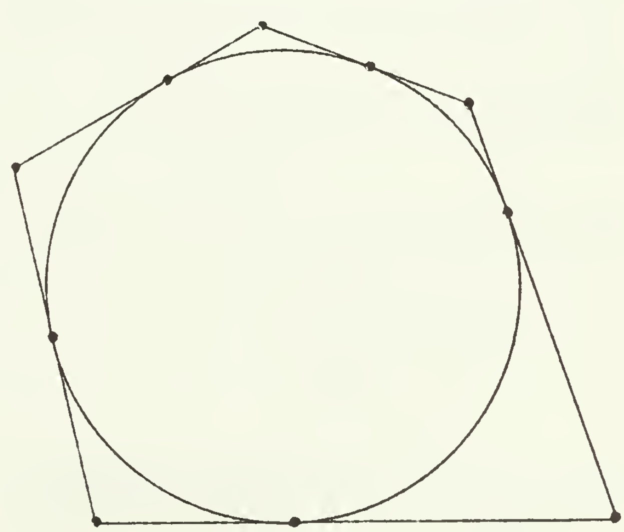
If the sides of a polygon are tangent to a circle then the polygon is said to be circumscribed about the circle, and the circle is said to be inscribed in the polygon.

An inscribed polygon



a polygon inscribed in a circle
a circle circumscribed about a polygon

A circumscribed polygon



a polygon circumscribed about a circle
a circle inscribed in a polygon

EXERCISES

A. 1. Draw a rectangle. Draw the circle which circumscribes the rectangle.

2. Draw another rectangle. Draw the circle which is inscribed in the rectangle.

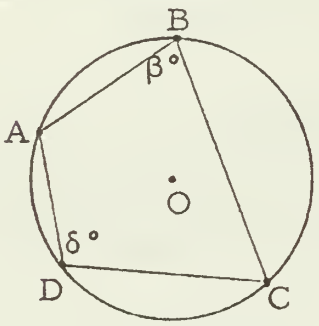
3. Inscribe a square in a given circle.

4. Inscribe an isosceles trapezoid in a given circle.

5. Circumscribe a square about a given circle.

* * *

Example.



Hypothesis: ABCD is inscribed in a circle with center O, $\overset{\frown}{ADC}$ is a minor arc

Conclusion: $\beta + \delta = 180$

Plan. Use the idea that the measures of $\overset{\frown}{ADC}$ and $\overset{\frown}{ABC}$ add up to 360.

Solution.

(1)	ABCD is an inscribed polygon	[Hypothesis]
(2)	$\angle B$ is an inscribed angle	[(1); defs. of inscribed polygon and inscribed angle]
(3)	______________________	[theorem]
(4)	$\beta = \frac{1}{2} \cdot m(\overset{\frown}{ADC})$	[(2) and (3)]
(5)	$\delta = \frac{1}{2} \cdot m(\overset{\frown}{ABC})$	[Steps like (2) - (4)]
(6)	$\beta + \delta = \frac{1}{2}[m(\overset{\frown}{ADC}) + m(\overset{\frown}{ABC})]$	[(4) and (5)]
(7)	$\overset{\frown}{ADC}$ is a minor arc	[Hypothesis]
(8)	$m(\overset{\frown}{ADC}) = 360 - m(\overset{\frown}{ABC})$	[(7); def. of arc-measure]
(9)	$\beta + \delta = 180$	[(6) and (8)]

Note. This example shows that if one of a pair of opposite angles of an inscribed quadrilateral intercepts a minor arc then the angles are supplementary. Now, suppose neither of the pair intercepts a minor arc. Are the opposite angles supplementary in that case?

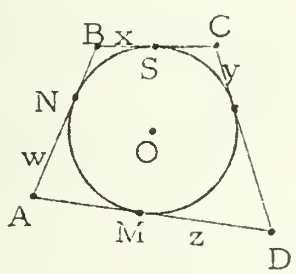
Thus, we have the following theorem:

> Theorem 10-33.
>
> The opposite angles of an inscribed quadrilateral are supplementary.

* * *

B. 1. Show that a parallelogram inscribed in a circle is a rectangle. [A rhombus inscribed in a circle is a ___?___]

2. Show that a trapezoid inscribed in a circle is isosceles.

3. 

Hypothesis: A circle with center O is inscribed in quadrilateral ABCD

Conclusion: AD + BC = AB + DC

4. Suppose that a circle is inscribed in a parallelogram. Prove that the parallelogram is a rhombus.

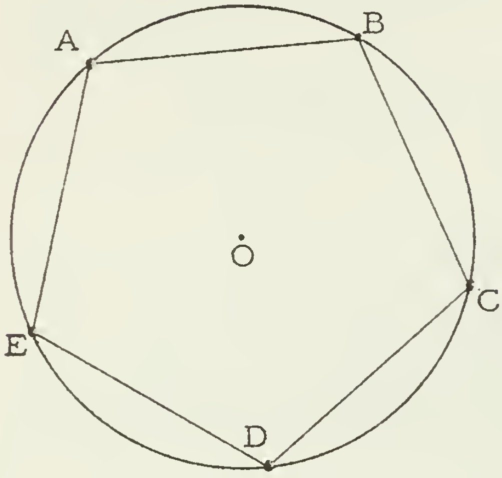
5. Show that a necessary and sufficient condition that a parallelogram have an incenter is that it be a rhombus.

6. A necessary and sufficient condition that a parallelogram have a circumcenter is that it be a ___?___. Prove it.

REGULAR POLYGONS AND CIRCLES

You have shown that a rhombus inscribed in a circle is a square. Now, of course, there are some rhombuses which are not squares. So, only certain rhombuses can be inscribed in a circle--just those which are equiangular. So, if an equilateral quadrilateral is inscribed in a circle, it is a regular quadrilateral.

Now, is this the case for pentagons, hexagons, etc? Let's look into the case for pentagons.



Suppose ABCDE is an equilateral pentagon inscribed in a circle with center O.

Is it regular? To show that it is, all we need show is that it is equiangular [Why?].

Consider $\angle A$ and $\angle B$. $\angle A$ is inscribed in $\overset{\frown}{EAB}$, and $\angle B$ is inscribed in $\overset{\frown}{ABC}$. But, $\overset{\frown}{EAB} = \overset{\frown}{EA} \cup \overset{\frown}{AB}$. So, $m(\overset{\frown}{EAB}) = m(\overset{\frown}{EA}) + m(\overset{\frown}{AB})$ [Why?]. Since $\overset{\frown}{EA} \cong \overset{\frown}{AB}$ [Why?], it follows that $m(\overset{\frown}{EAB}) = 2 \cdot m(\overset{\frown}{AB})$.

Similarly, we can show that $m(\overset{\frown}{ABC}) = 2 \cdot m(\overset{\frown}{AB})$. So, $\overset{\frown}{EAB} \cong \overset{\frown}{ABC}$. Hence [by what theorem?], $\angle A \cong \angle B$.

Therefore, each two adjacent angles of the pentagon are congruent. So, all of the angles of the pentagon are congruent.

How did we make use of the hypothesis that the figure is a pentagon? Clearly, the same argument shows that

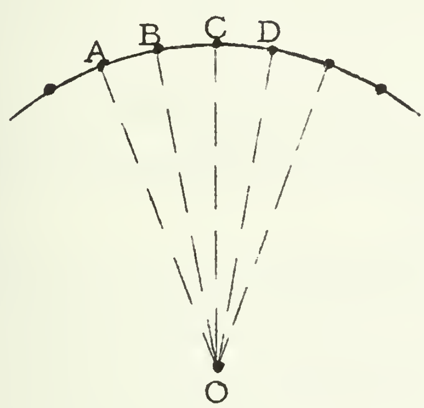
> <u>Theorem 10-34.</u>
>
> An equilateral polygon inscribed in a circle is regular.

[Investigate the analogous problem concerning equiangular polygons inscribed in a circle.]

If you are given a polygon, is it possible to circumscribe a circle about it? In an earlier section you proved this was the case for three-sided polygons when you proved that each triangle has a <u>circumcircle.</u>

Consider, now, the case of quadrilaterals. Certainly, some quadrilaterals have circumcircles. We have shown in proving Theorem 10-33 that if a quadrilateral is inscribed in a circle, then its opposite angles are supplementary. That is, if a quadrilateral has a circumcircle, its opposite angles are supplementary. Consequently, if a pair of opposite angles of a quadrilateral are not supplementary, the quadrilateral does not have a circumcircle.

In general, a sufficient condition [but not a necessary one--why not!] for a polygon to have a circumcircle is that it be a regular polygon. Let's look at a proof of this.



Hypothesis: ABCD...is a regular polygon

Conclusion: A, B, C, D,...are concyclic

Since A, B, and C are noncollinear, A, B, and C determine a circle. Let O be its center. We shall now prove that D belongs to this circle by showing that OD = OA.

By hypothesis, ABCD ... is a regular polygon. So, BA = BC. Since O is the circumcenter of $\triangle ABC$, OA = OC. So, $\overleftrightarrow{BO}$ is the perpendicular bisector of $\overline{AC}$. But, $\triangle ABC$ is isosceles. So, $\overrightarrow{BO}$ is the bisector of $\angle ABC$, and $m(\angle ABO) = \frac{1}{2} \cdot m(\angle ABC)$.

Similarly, $m(\angle DCO) = \frac{1}{2} \cdot m(\angle BCD)$. By hypothesis, $\angle ABC \cong \angle BCD$. So, $\angle ABO \cong \angle DCO$.

Now, OB = OC, and, by hypothesis, BA = CD. So, OBA $\longleftrightarrow$ OCD is a congruence [Why?]. Hence, OD = OA.

So, we have shown that each four adjacent vertices of a regular polygon are concyclic. It is very easy to show that each five adjacent vertices of a regular polygon are concyclic.

By using a theorem about numbers which you will study in a later unit, we could use the reasoning in the foregoing paragraphs to prove that a regular polygon of n sides has a unique circumcircle.

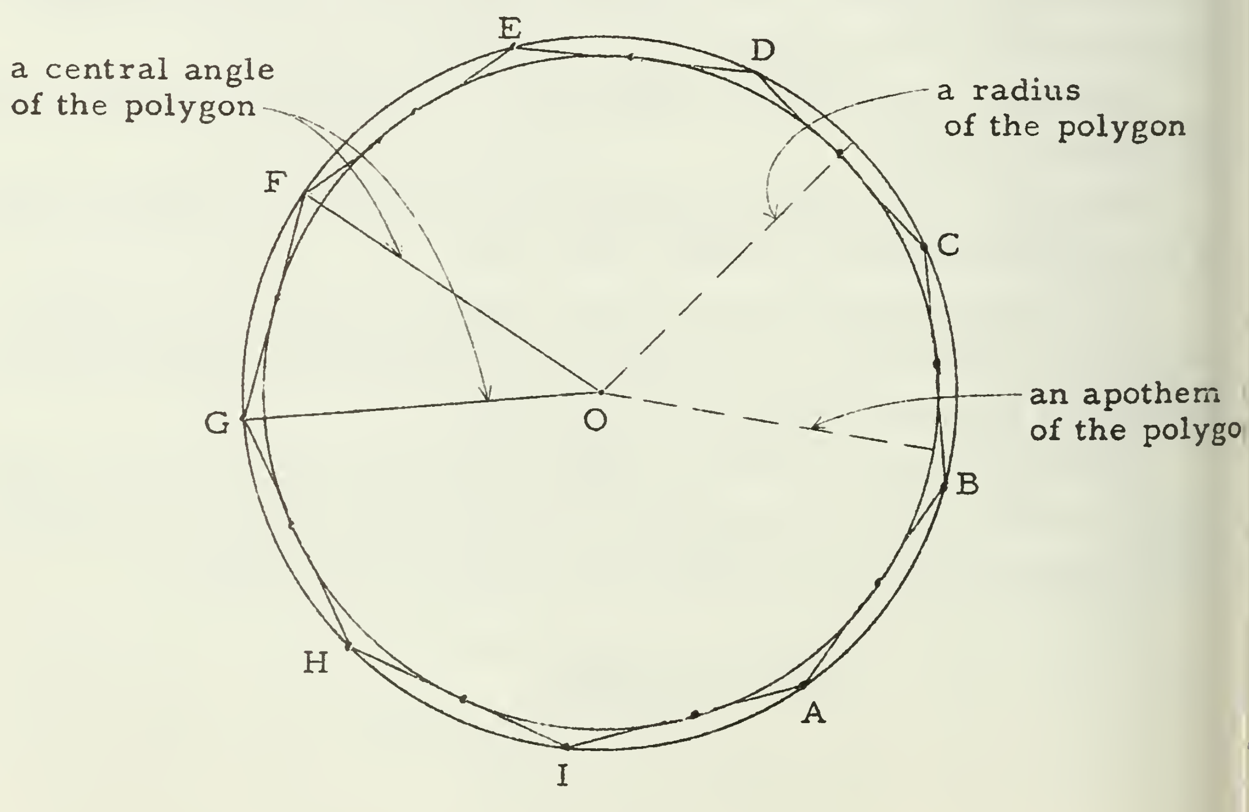
> Theorem 10-35.
>
> A regular polygon has a unique circumcircle.

Now, consider the center of a regular polygon's circumcircle. Let O be this point, and let $\overleftrightarrow{AB}$ be a side of the regular polygon. If M is the midpoint of $\overleftrightarrow{AB}$ then $\overleftrightarrow{OM}$ is perpendicular to $\overleftrightarrow{AB}$ [Why?]. Consider the circle with center O and radius OM. Is $\overleftrightarrow{AB}$ tangent to this circle? Is each side of the regular polygon tangent to the circle? [Explain.]

The circle whose center is O and to which each side of the regular polygon is tangent is, of course, the circle inscribed in the polygon. We shall call it <u>the incircle</u> of the polygon.

Theorem 10-36.
A regular polygon has a unique incircle.

The common center of a regular polygon's circumcircle and incircle is called <u>the center</u> of the polygon. A radius [radial segment] of the circumcircle is called <u>a radius</u> of the regular polygon, and a radius of the incircle is called <u>an apothem</u> of the polygon. An angle whose vertex is the center of a regular polygon and whose sides contain adjacent vertices is called <u>a central angle</u> of the polygon.



* * *

EXERCISES

A. Find the radius and the apothem for each figure.

1. An equilateral triangle with altitude 8.
2. A square with side 30.
3. An equiangular triangle circumscribed about a circle with diameter 10.
4. An equiangular triangle with side 6.

B. Prove each of the following:

1. An equilateral polygon circumscribed about a circle is regular.
2. An apothem which intersects a side of its regular polygon is perpendicular to that side at its midpoint.
3. A radius of a regular polygon drawn to a vertex is a subset of the bisector of the angle of the polygon with that vertex.
4. A central angle of a regular n-gon is an angle of $\frac{360°}{n}$.
5. A central angle of a regular polygon is congruent to any exterior angle of the polygon.
6. If an equilateral triangle is inscribed in a circle and another equilateral triangle is circumscribed about the circle then the ratio of their perimeters is 1 : 2.

C. 1. Draw a circle of radius 4.

(a) Inscribe a square.
(b) Find the perimeter of the square.
(c) Find the apothem of the square.
(d) Inscribe a regular octagon in the circle.
(e) Find the perimeter of the octagon.
(f) Find the apothem of the octagon.

2. Find the apothem and the perimeter of a regular ten-sided polygon inscribed in a circle of radius 4.

3. Find the apothem and the perimeter of a regular 16-sided polygon inscribed in a circle of radius 4. Do the same for a regular 32-sided polygon.

4. Find the perimeter of a regular 32-sided polygon circumscribed about a circle of radius 4.

5. Show that the perimeter of a regular n-gon inscribed in a circle is less than the perimeter of a 2n-sided regular polygon inscribed in that circle.

6. Show that the perimeter of an n-sided regular polygon inscribed in a circle is less than the perimeter of an n-sided regular polygon circumscribed about that circle.

THE CIRCUMFERENCE OF A CIRCLE

If you inscribe two regular polygons in a circle of radius 4 and compute their perimeters, you find that the polygon with the larger number of sides has the larger perimeter. If p_n is the perimeter of a regular polygon of n sides inscribed in a circle then

$$(1) \qquad p_3 < p_4 < p_5 < \ldots < p_k < p_{k+1} < \ldots .$$

If you circumscribe the circle of radius 4 with two regular polygons, the polygon with the larger number of sides has the smaller perimeter. If p'_n is the perimeter of a regular polygon of n sides circumscribed about this circle then

$$(2) \qquad p'_3 > p'_4 > p'_5 > \ldots > p'_k > p'_{k+1} > \ldots .$$

Furthermore, as you showed in a previous exercise, for each regular polygon of n sides,

$$p_n < p'_n .$$

This means that although the numbers in the sequence $p_3, p_4, p_5, \ldots, p_k, p_{k+1}, \ldots$ are increasing, there is a number ["an upper bound"] which is larger than any of them. In fact, there are lots of upper bounds which are larger than any of the perimeters. And, it can be proved that there is a smallest number among these upper bounds. Let c be this least upper bound.

Also, there are lots of numbers ["lower bounds"] which are smaller than any of the numbers in the sequence $p'_3, p'_4, p'_5, \ldots$. It can be proved that there is a largest number among these lower bounds. Let c' be this greatest lower bound.

Finally, it can be proved that $c = c'$. This common bound, for a circle of radius 4, is a number k. If you worked this problem for a circle of radius 7, the common bound would be another number k'. And, the ratio of k to the diameter 8 would be the same as the ratio of k' to the diameter 14. In fact, this is the ratio you would get for every circle. This ratio is the number pi (π).

So, for each circle of radius r, $(2r)\pi$ is the common bound of the sequences of perimeters of inscribed and circumscribed regular polygons. This number, $(2r)\pi$, is called <u>the circumference of the circle.</u>

<u>Theorem 10-37.</u>

The circumference of a circle is the product of its radius and 2π.

$$c = 2\pi r$$

The number π is an irrational number. A rational approximation correct to 8 decimal places in 3.14159265.

<u>Example 1.</u> Compute the circumference of a circle with radius 5.

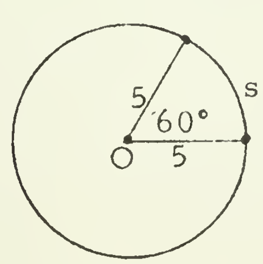
<u>Solution.</u>

$$\begin{aligned} c &= 2\pi r \\ &= 2\pi \cdot 5 \\ &= 10\pi \end{aligned}$$

The circumference is 10π.

<u>Example 2.</u> Compute the length-measure of an arc of 60° of a circle with radius 5.

<u>Solution.</u>



$$\begin{aligned} s &= \frac{60}{360} \cdot 2\pi r \\ &= \frac{1}{6} \cdot 2\pi \cdot 5 \\ &= \frac{5}{3}\pi \end{aligned}$$

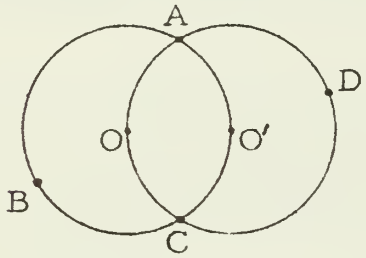
EXERCISES

1. Find the circumference of a circle with radius 2.

2. Find the circumference of a circle with diameter 1.

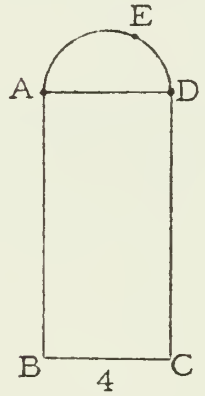
3. Complete the table:

degree-measure of $\overset{\frown}{ABC}$	45	180			60			
radius of $\overset{\frown}{ABC}$	12	6	10	5		7	5	1
length-measure of $\overset{\frown}{ABC}$			4π	6π	6π	15π	5	$\frac{\pi}{6}$

4. 

Given: two circles each with radius 10, and each containing the other's center

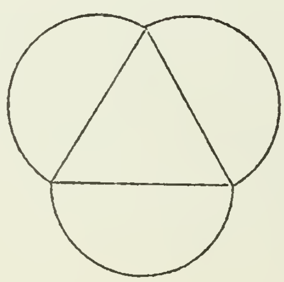
Find: length-measure of $\overset{\frown}{ABC} \cup \overset{\frown}{CDA}$

5. 

Given: $\overset{\frown}{AED}$ is a semicircle, ABCD is a rectangle

Find: perimeter of AEDCBA

6. Draw semicircles with sides of an equilateral triangle as diameters. If the perimeter of the triangle is k, what is the distance around the figure?



SUMMARY OF SECTION 6.10

Notation and terminology

Theorems

10-1. For each circle c with radius r and each line ℓ at a distance d from the center of c, $\ell \cap c$ consists of two points if and only if $d < r$; $\ell \cap c$ consists of one point if and only if $d = r$; and $\ell \cap c = \emptyset$ if and only if $d > r$.

10-2. A line is a tangent to a circle at a given point if and only if it contains the point and is perpendicular to the radius at that point.

10-3. The radius perpendicular to a chord bisects the chord.

10-4. Two chords of a circle are congruent if and only if they are equidistant from the center.

10-5. Two tangents to a circle are parallel if and only if the points of tangency are the end points of a diameter.

10-6. The perpendicular bisector of a chord contains the center of the circle.

10-7. The line containing the center of a circle and the midpoint of a chord of the circle which is not a diameter is perpendicular to the chord.

10-8. The segments joining the point of intersection of two tangents to the points of tangency are congruent.

10-9. The diameters of a circle are congruent.

10-10. Three noncollinear points determine a circle.

10-11. Each triangle has a unique incircle.

10-12. For each two circles c_1 and c_2 with radii r and s, respectively, such that $r \geq s$, and with centers a distance $d \neq 0$ apart, $c_1 \cap c_2$ consists of two points if and only if $r - s < d < r + s$; $c_1 \cap c_2$ consists of one point if and only if $d = r - s$ or $d = r + s$; and $c_1 \cap c_2$ is the empty set if and only if $d < r - s$ or $d > r + s$.

10-13. If two circles intersect in two points, the line of centers is the perpendicular bisector of the common chord.

10-14. For all nonzero numbers of arithmetic, x, y, and z, if $x \geq y$ then there is a triangle whose side measures are x, y, and z, respectively, if and only if $x - y < z < x + y$.

10-15. Arcs of the same or congruent circles which are intercepted by congruent central angles are congruent.

10-16. Semicircles of the same or congruent circles are congruent.

10-17. Major arcs of the same or congruent circles are congruent if and only if the related minor arcs are congruent.

10-18. Central angles of the same or congruent circles which intercept congruent arcs are congruent.

10-19. Minor arcs of the same or congruent circles are congruent if and only if their chords are congruent.

10-20. A diameter perpendicular to a chord of the circle bisects both arcs whose end points are those of the chord.

10-21. The sum of the measures of two concyclic arcs which have only an end point in common is the measure of their union.

10-22. The measure of an inscribed angle is half the measure of its intercepted arc.

10-23. The measure of an angle one of whose sides is tangent to a circle at the vertex and whose other side intersects the circle is half the measure of the intercepted arc.

10-24. The measure of an angle whose vertex is an internal point of a circle is half the sum of the measures of the arcs intercepted by the angle and its vertical angle.

10-25. The measure of an angle whose vertex is an external point of a circle and whose sides intersect the circle is half the difference of the intercepted arcs.

10-26. An angle whose sides are tangent to a circle is a supplement of the central angle whose sides contain the points of tangency.

10-27. Parallel lines which intersect a circle cut off congruent arcs.

10-28. Angles inscribed in the same arc or in congruent arcs are congruent.

10-29. An angle inscribed in a semicircle is a right angle.

10-30. If two chords of a circle intersect, the point of intersection divides each chord into segments such that the product of the measures of the segments of one chord is the product of the measures of the segments of the other.

10-31. If two secants of a circle intersect in an external point, the product of the distances between the external point and the points of intersection on one secant is the product of the distances between the external point and the points of intersection on the other secant.

10-32. If a secant and a tangent of a circle intersect in an external point, then the tangent segment is the mean proportional between the secant segment and the external segment of the secant.

10-33. The opposite angles of an inscribed quadrilateral are supplementary.

10-34. An equilateral polygon inscribed in a circle is regular.

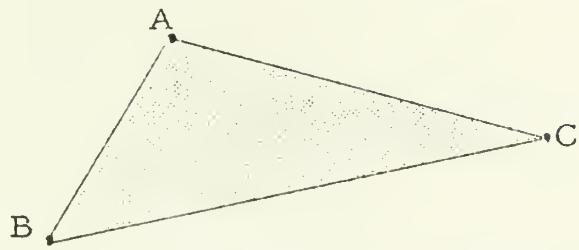
10-35. A regular polygon has a unique circumcircle.

10-36. A regular polygon has a unique incircle.

10-37. The circumference of a circle is the product of its radius and 2π. $c = 2\pi r$.

[Supplementary exercises are on page 6-439.]

6.11 <u>Measures of regions.</u>--Here is a picture of a triangular region:

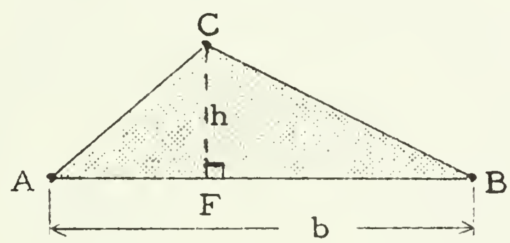


<u>A triangular region</u> is the union of a triangle and its interior. The one pictured above is designated by the symbol:

$$\blacktriangle ABC$$

[Read this as 'triangular region ABC'.] The triangle $\triangle ABC$ is the <u>boundary</u> of $\blacktriangle ABC$.

In this section, we shall learn how to assign measures to regions. Such measures are called <u>area-measures</u>. The system of area-measures will be denoted by 'K'. So, for example, '$K(\blacktriangle ABC)$' stands for the area-measure of $\blacktriangle ABC$, and we read it as 'K of the triangular region ABC' or as 'the area-measure of the triangular region ABC', or as 'the measure of the triangular region ABC'.



In earlier grades you have learned that the area-measure of $\blacktriangle ABC$ is given by the formula:

$$(*) \quad K(\blacktriangle ABC) = \frac{1}{2}bh,$$

where b is the measure of "the" base $\overline{AB}$ of the triangle, and h is the measure of the altitude $\overline{CF}$ of $\triangle ABC$ from the opposite vertex C. Now, having become more critical, you might wonder about this definition. Why, for example, should $\overline{AB}$ be called <u>the</u> base of $\triangle ABC$? Certainly, each of the three sides is as good to use as base as any other. But, this raises the question of whether (*) will give the same result if b is the measure of, say $\overline{BC}$, and h is the measure of the altitude of $\triangle ABC$ from A. Fortunately, you have already shown that this is the case--the product of the measure of any side of a triangle by the measure of the corresponding altitude is the same as the product of the measure

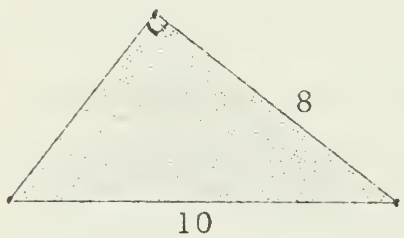
of either of the other sides by the measure of <u>its</u> corresponding altitude [page 6-214]. So, we can define the area-measure of a triangular region to be half the product of the measure of <u>any</u> side by the measure of the altitude from the opposite vertex.

EXERCISES

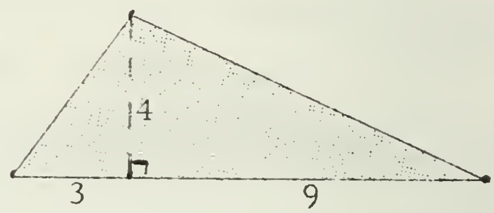
Use (*) in solving the following problems.

<u>A.</u> Compute the area-measure of the triangular regions.

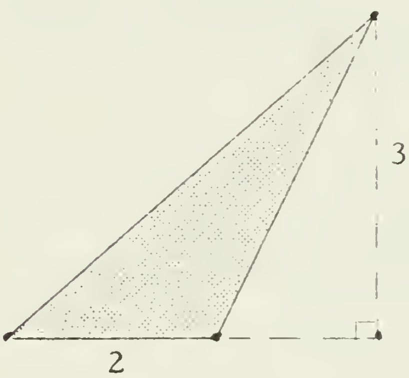
1.



2.



3.



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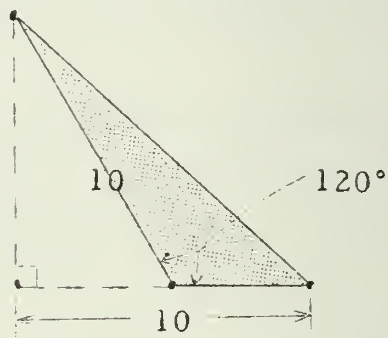
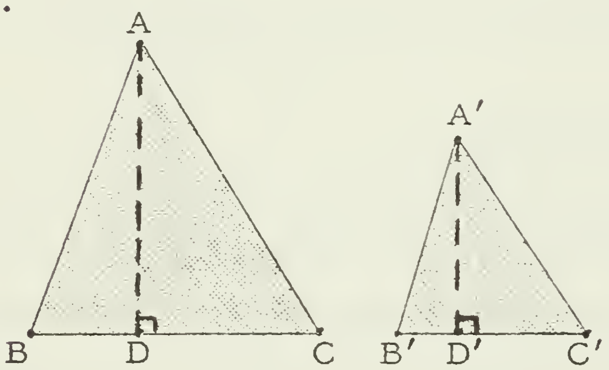


<u>B.</u> 1.



<u>Hypothesis:</u> $AD = A'D'$,

$\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$,

$\overleftrightarrow{A'D'} \perp \overleftrightarrow{B'C'}$,

$\dfrac{B'C'}{BC} = r$

<u>Conclusion:</u> $\dfrac{K(\blacktriangle A'B'C')}{K(\blacktriangle ABC)} = r$

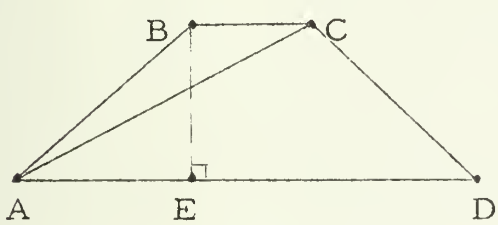
2. Show that if two triangles are congruent then the triangular regions which they bound have the same measure.

3. Is the converse of the conditional sentence in Exercise 2 a theorem?

4. [Refer to the figure for Exercise 1.] Show that if the bases are congruent and the ratio of the altitude $\overleftrightarrow{A'D'}$ to $\overleftrightarrow{AD}$ is r then $K(\blacktriangle A'B'C')/K(\blacktriangle ABC) = r$.

5. Given $\triangle ABC$. Describe the set of all points P such that $K(\blacktriangle PAB) = K(\blacktriangle ABC)$.

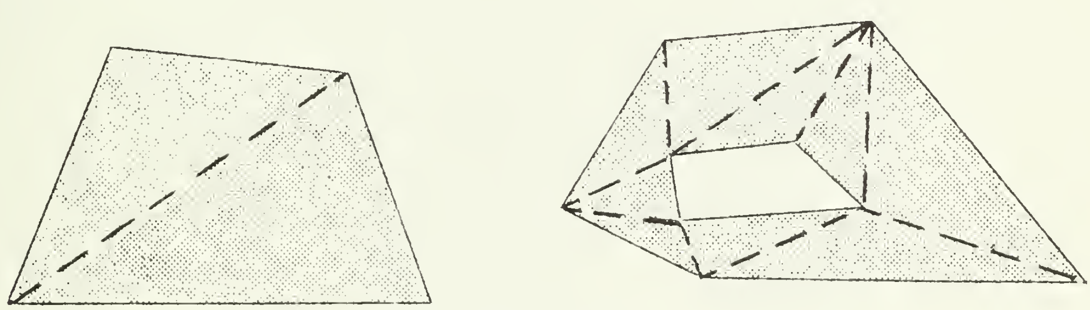
6. Consider the quadrilateral ABCD and a diagonal $\overleftrightarrow{AC}$. If the altitude of $\triangle ABC$ from B is twice as long as the altitude of $\triangle DAC$ from D, what is the ratio of $K(\blacktriangle ABC)$ to $K(\blacktriangle DAC)$?

7. 

ABCD is a trapezoid, and $\overleftrightarrow{BE} \perp \overleftrightarrow{AD}$. If BC = 3, BE = 4, and AD = 7, what is the ratio of $K(\blacktriangle ABC)$ to $K(\blacktriangle ACD)$?

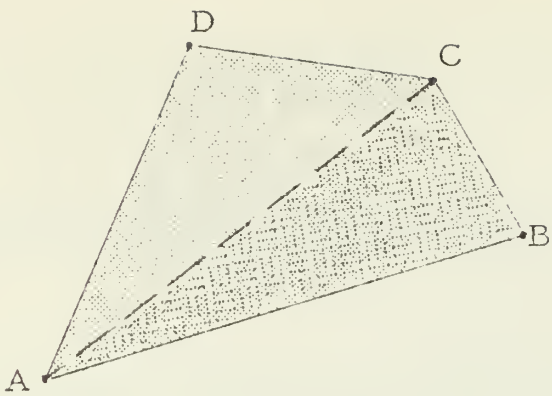
POLYGONAL REGIONS

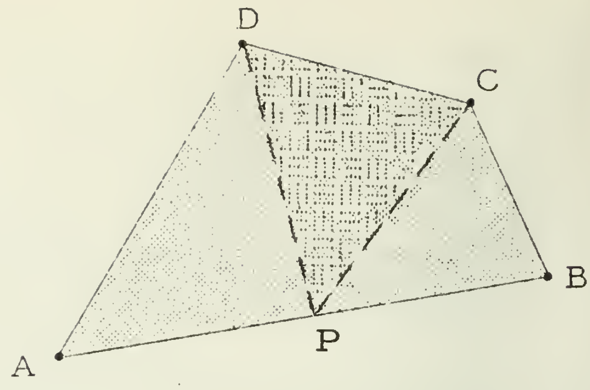
A polygonal region is a set which is the union of triangular regions such that the intersection of any two of them is a subset of the boundary of each. [Such triangular regions are called the component triangular regions of the polygonal region.]



polygonal regions

It is natural to define the area-measure of a polygonal region as the sum of the area-measures of the triangular regions whose union it is. But, here we run into the same sort of trouble that we had with the definition of the area-measure of a triangle. There, the phrase 'the base' caused trouble which we overcame by showing that we get the same number as area-measure of a triangular region, no matter which side we use as base. Here, the phrase 'the triangular regions whose union it is' causes the same kind of trouble.





Will we get the same result when we add K(▲ABC) and K(▲CDA) as we do when we add K(▲APD), K(▲CPB), and K(▲CDP)? Fortunately, it can be proved that this is the case. No matter how a polygonal region is "cut up" into triangular regions, the sum of the area-measures of the triangular regions is the same. This is hard to prove, so we shall accept it without proof. In view of this we can define the area-measure of a polygonal region as the sum of the area-measures of <u>any</u> triangular regions into which it can be cut up.

Consequently, we can base our discussion of area-measure on two axioms.

<u>Axiom I.</u>

The area-measure of a triangular region is half the product of the measure of any of its sides by the measure of the altitude from the opposite vertex of the triangle.

<u>Axiom J.</u>

The area-measure of a polygonal region is the sum of the area-measures of any set of component triangular regions into which it can be cut.

Consider the polygonal region bounded by rectangle ABCD. We use

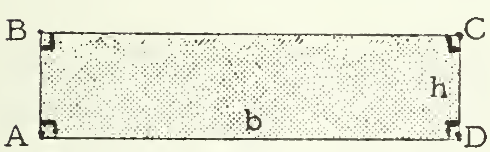
Axiom I and Axiom J to compute the area-measure of this polygonal

region. The diagonal $\overleftrightarrow{AC}$ cuts the region ABCD into two triangular regions ▲ABC and ▲ADC. Since $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$ [Why?], $\overleftrightarrow{AB}$ is an altitude of $\triangle ABC$ from A and $\overleftrightarrow{BC}$ is the corresponding base. Hence, K(▲ABC) $= \frac{1}{2} \cdot BC \cdot AB$ [Axiom I]. Similarly, K(▲ADC) $= \frac{1}{2} \cdot AD \cdot CD$. So, by Axiom J, K(▬ABCD) = K(▲ABC) + K(▲ADC) $= \frac{1}{2} \cdot BC \cdot AB + \frac{1}{2} \cdot AD \cdot CD$. But, AB = CD and BC = AD [Why?]. Thus, K(▬ABCD) $= \frac{1}{2} \cdot BC \cdot AB + \frac{1}{2} \cdot BC \cdot AB = BC \cdot AB$.

So, we have proved the following theorem:

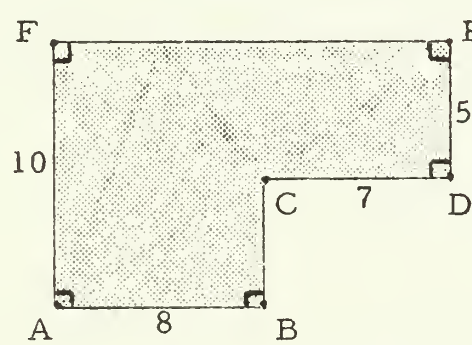
> Theorem 11-1.
>
> The area-measure of a rectangular region is equal to the product of the measures of a pair of adjacent sides of the boundary of the region.

The result stated in Theorem 11-1 can be written as:

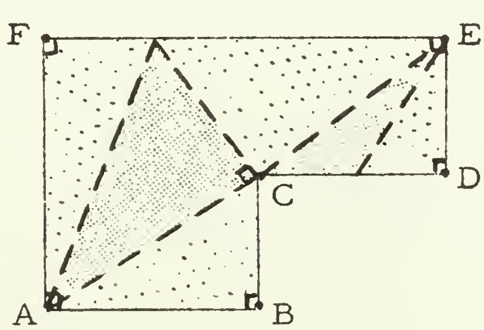


K(▬ABCD) = bh

What is the measure of the polygonal region ABCDEF?



There are many ways to cut the region ABCDEF into component triangular regions. For each such cutting, you can use Axioms I and J to find the area-measure of the region. Here is one way:



You probably did not solve the problem by cutting the regions as shown on page 6-339. [What is the least number of component triangular regions into which you can cut the region ABCDEF?] In fact, you probably recognized that the region ABCDEF is the union of two rectangular regions. If so, you computed the area-measure of region ABCDEF by finding the sum of the area-measures of those two rectangular regions. If you solved the problem in this way, then you made use of a theorem which is a consequence of Axiom J and Introduction Axioms.

> Theorem 11-2.
>
> The area-measure of the union of two polygonal regions whose intersection is contained in the boundary of each of them is the sum of the area-measures of the given polygonal regions.

Theorem 11-2 gives us a general method for assigning area-measures to polygonal regions. It consists of cutting the regions into simpler ones, and summing the area-measures of the component regions. The simplicity of the method depends upon the manner in which you cut the region. For example, consider finding a formula for the area-measure of a regular hexagon [that is, a region whose boundary is a regular hexagon], each of whose sides has measure s. Here are some of the ways of cutting the region. It is probably easiest to use the one

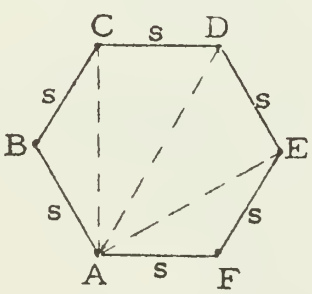


Figure 1

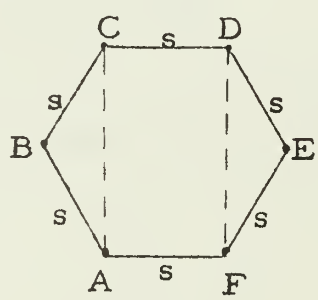


Figure 2

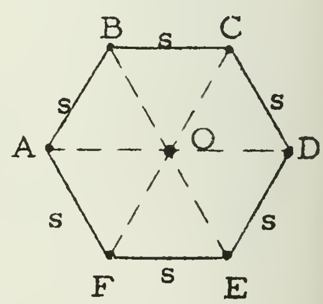


Figure 3

in Figure 3, since this involves computing the area-measure of, say, $\blacktriangle$ODC, and multiplying it by 6. [O is the center of the circumscribed circle.] Since $\triangle$ODC is isosceles, and since the degree-measure of $\angle$COD is 60, the measure of the altitude from O of $\triangle$ODC is $\frac{s\sqrt{3}}{2}$. So,

the area-measure of $\blacktriangle$ODC is

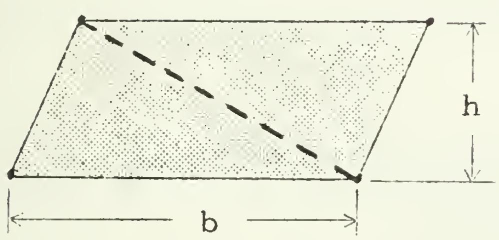
$$\left[\frac{1}{2}\left(\frac{s\sqrt{3}}{2}\right)(s)\right], \text{ or } \frac{s^2\sqrt{3}}{4}.$$

Hence, the area-measure of the regular hexagon is $\frac{3s^2\sqrt{3}}{2}$.

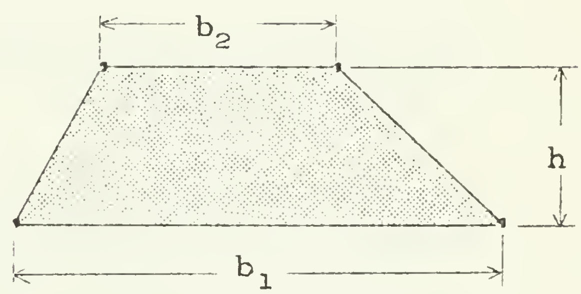
EXERCISES

A. Derive formulas for the area-measures of the polygonal regions given below.

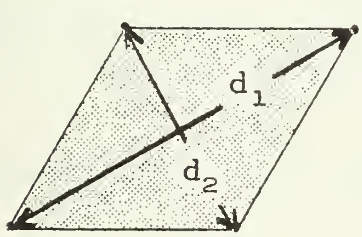
1. Parallelogram



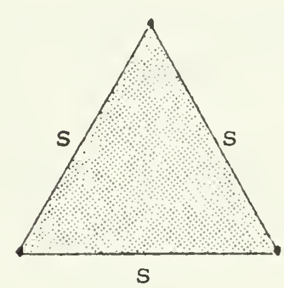
2. Trapezoid



3. Rhombus



4. Equilateral triangle



B. Solve the following problems.

1. What is the cost of tiling a rectangular floor 18 feet by 24 feet if each tile is a square 9 inches on each side and costs 13 cents?

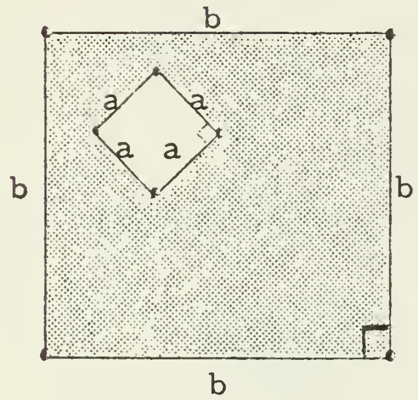
2. Find the area-measure of a square whose diagonal's measure is 15.

3. Find the area of a square whose diagonal is 15 inches long.

4. The measures of two sides of a parallelogram are 15 and 25, and the measure of one of its angles is 60. What is the area-measure?

5. A rectangle is inscribed in a circle whose diameter is 50 feet. If the width of the rectangle is 20 feet, what is its area?

6.



Find a formula for the area-measure of the shaded region.

7. Prove that the median of a triangle divides it into two triangles with the same measure.

8. Prove that the diagonals of a parallelogram divide it into four triangles of the same area-measure.

9. The measure of a diagonal of a rhombus is 6 and the measure of each side is 10. Find its area-measure.

10. The area of a square is 100 square inches. Find the length of a side.

11. Find the area of an equilateral triangle whose altitude is $5\sqrt{3}$ inches long.

12. Find the measure of a side of a square whose area-measure is the sum of the area-measures of squares whose sides measure 5 and 12, respectively.

13. The measure of $\overleftrightarrow{AB}$ is 9. Describe the set of points P such that the area-measure of ▲PAB is 18.

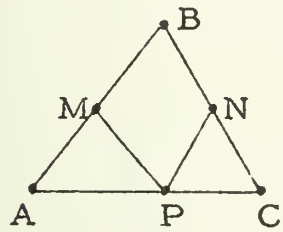
14. The area-measure of a 30-60-90 triangle is $10\sqrt{3}$. Find the measures of the three sides.

15. The area-measure of a trapezoid is 126, one base measures 12, and the altitude measures 9. What is the measure of the other base?

16. Find the area-measure of a square inscribed in a circle whose radius is 6.

17. One leg of a right triangle measures 10 and the altitude upon the hypotenuse measures 6. What is the area-measure of the triangle?

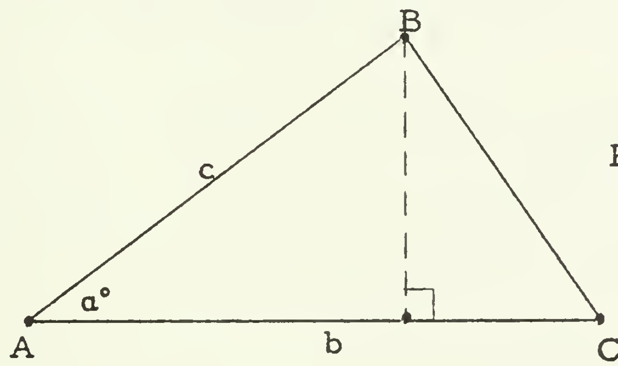
18. The ratio of the legs of a right triangle is $\frac{3}{4}$. The altitude upon the hypotenuse measures 16. Find the area-measure of the triangle.

19. 

Hypothesis: M and N are midpoints, $P \in \overline{AC}$

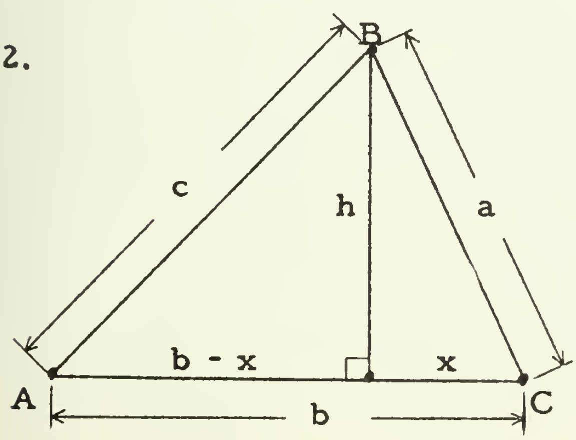
Conclusion: the area-measure of MPNB is half that of ▲ABC

20. Derive a formula for the area-measure of a triangle in terms of the measure of one of its acute angles and the measures of the including sides.



K(▲ABC) = ________

21. Derive a formula for the area-measure of a regular pentagon. [Your formula may include a 'sin' or a 'cos'.]

22. 

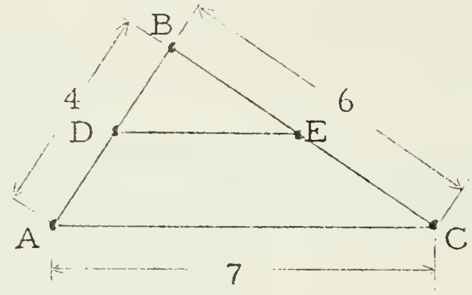
Justify the formula:

$K(\blacktriangle ABC) = \sqrt{s(s-a)(s-b)(s-c)}$

where s is the semi-perimeter, that is, $s = \frac{a+b+c}{2}$.

EXPLORATION EXERCISES

Consider the triangle $\triangle ABC$, the measures of whose sides are 4, 6, and 7. Suppose D and E are the midpoints of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CB}$, respectively.



1. Prove that $ABC \longleftrightarrow DBE$ is a similarity.

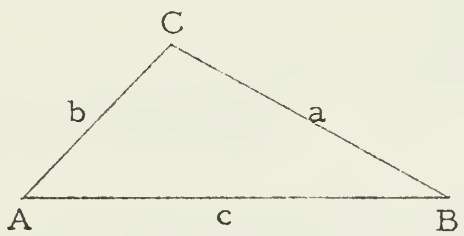
2. Compute the perimeters of $\triangle ABC$ and $\triangle DBE$, and compare them.

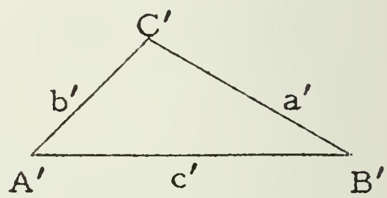
3. Compare the measures of the altitudes from B of $\triangle ABC$ and $\triangle DBE$.

4. Compare the area-measures of $\blacktriangle ABC$ and $\blacktriangle DBE$.

AREAS AND SIMILAR TRIANGLES

Consider $\triangle ABC$ and $\triangle A'B'C'$. Suppose that $ABC \longleftrightarrow A'B'C'$ is a





similarity. Then, by definition,

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}.$$

These last equations tell you that there is a number s such that

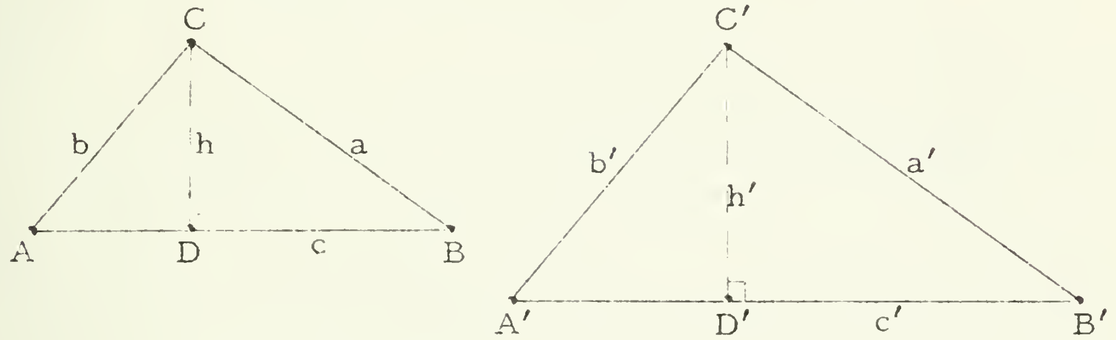
$$a = sa', \quad b = sb', \quad \text{and } c = sc'.$$

The number s is called <u>the ratio of similitude of $\triangle ABC$ to $\triangle A'B'C'$</u>.

It is easy to see that the ratio of the perimeter of $\triangle ABC$ to the perimeter of $\triangle A'B'C'$ is s. For if p and p′ are the perimeters of $\triangle ABC$ and $\triangle A'B'C'$, respectively, then

$$\frac{p}{p'} = \frac{a + b + c}{a' + b' + c'} = \frac{sa' + sb' + sc'}{a' + b' + c'} = s \quad [\text{Why?}].$$

Suppose $ABC \leftrightarrow A'B'C'$ is a similarity, and k and k' are the area-measures of the triangles. What do you think is the ratio of k to k'? To answer this question, we compute the area-measures of the triangles.



(1) $k = \frac{1}{2}hc$ (2) $k' = \frac{1}{2}h'c'$

$$(3)\quad \frac{k}{k'} = \frac{\frac{1}{2}hc}{\frac{1}{2}h'c'} = \frac{hc}{h'c'}$$

Since, in the right triangles $\triangle ADC$ and $\triangle A'D'C'$, $\angle A \cong \angle A'$, the right triangles are similar. So,

$$\text{if } \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = s \text{ then } \frac{h}{h'} = s.$$

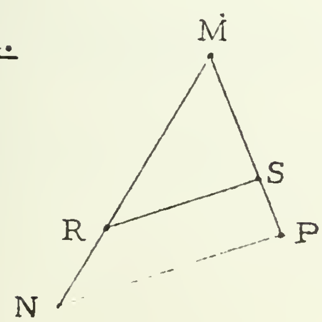
Hence, from (3) we have:

$$(4)\quad \frac{k}{k'} = \frac{h}{h'} \cdot \frac{c}{c'} = s \cdot s = s^2$$

We have shown that the ratio of the area-measures of two similar triangles is the square of the ratio of similitude.

EXERCISES

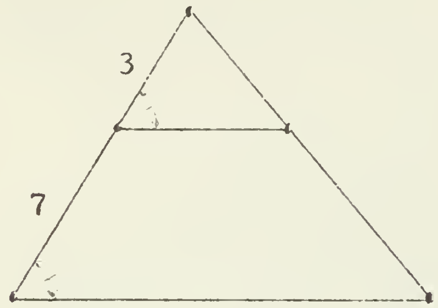
A.



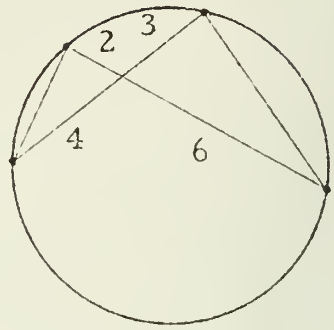
In $\triangle MNP$, the point R divides $\overleftrightarrow{MN}$ in the ratio 2 to 1, and the point S divides $\overleftrightarrow{MP}$ in the same ratio. What is the ratio of the area-measure of $\blacktriangle MRS$ to the area-measure of $\blacktriangle MNP$?

B. Compute the ratio of the area-measure of the smaller triangle to the area-measure of the larger triangle.

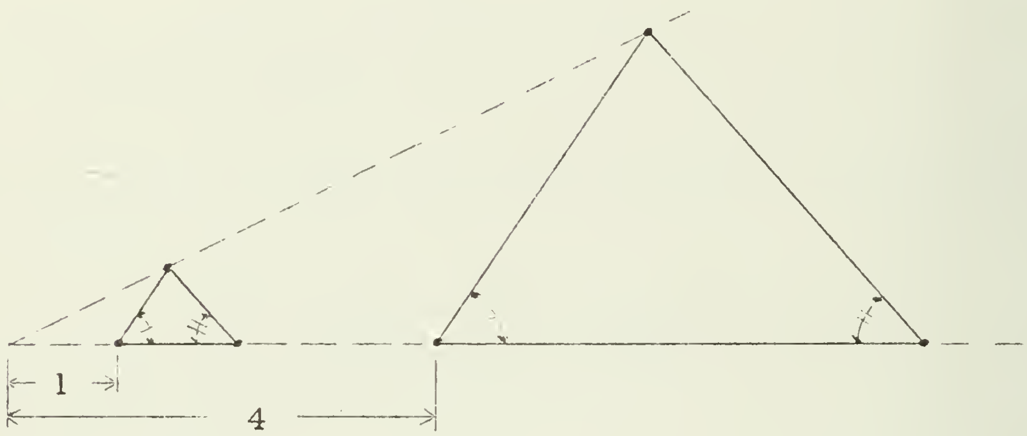
1.



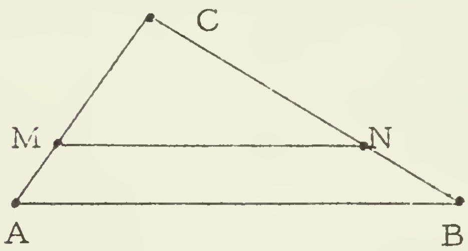
2.



3.



C.



If $\overleftrightarrow{MN} \parallel \overleftrightarrow{AB}$ and the area-measure of ▲MCN is half the area-measure of ▲ABC, what is the ratio of MC to MA?

D. We have said that similar polygons are polygons whose vertices can be matched in such a way that corresponding angles are congruent and corresponding sides are proportional.

1. Prove that each two squares are similar.

2. $ABCDE \longleftrightarrow A'B'C'D'E'$ is a similarity for two pentagons. Prove that $\triangle BED \sim \triangle B'E'D'$.

3. Prove the ratio of the area-measures of two similar quadrilaterals is the square of the ratio of similitude.

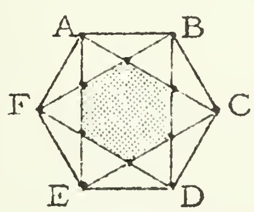
4. Generalize the theorem in Exercise 3.

E. Solve the following problems.

1. The measures of a pair of corresponding sides of two similar polygons are 3 and 7. What is the ratio of a pair of corresponding diagonals? What is the ratio of the perimeters? What is the ratio of their area-measures?

2. The area-measure of a triangle is 25 times the area-measure of a similar triangle. What is the ratio of similitude of the smaller triangle to the larger triangle?

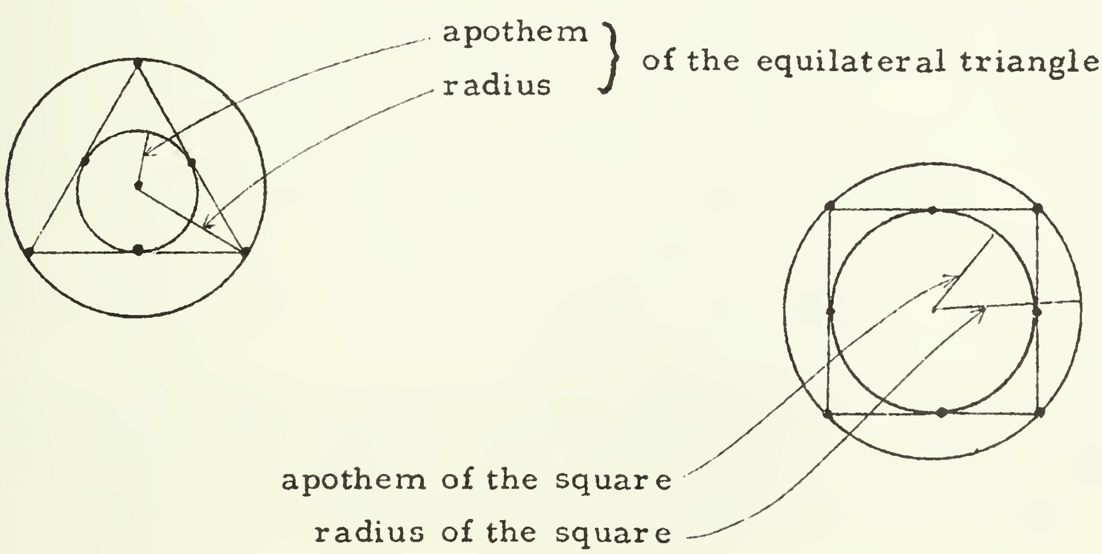
3. The difference between the area-measures of two squares is 825. If the ratio of a pair of corresponding sides is 4 to 7, what are the area-measures of the squares?

4. 

Show that the area-measure of the shaded region is a third of the area-measure of the regular hexagon ABCDEF.

AREA-MEASURE OF A CIRCULAR REGION

Recall that the radius of a regular polygon is the radius of its circumcircle and that the apothem of a regular polygon is the radius of its incircle.



EXERCISES

A. 1. Find the ratio of the apothem to the radius of an equilateral triangle.

2. Repeat (1) for a square.

3. Repeat (1) for a regular hexagon.

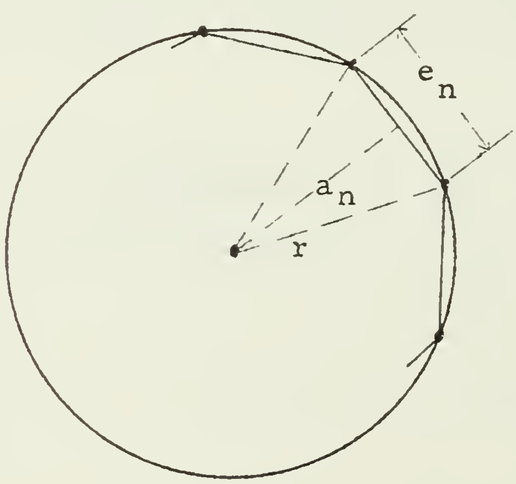
4. Prove that the area-measure of a regular polygon is half the product of its apothem and its perimeter.

* * *

In the last exercise you proved the following theorem:

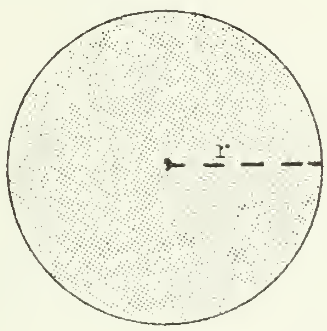
> Theorem 11-3.
>
> The area-measure of a regular polygonal region is half the product of the apothem and the perimeter of the boundary.

This theorem, as you probably suspect, leads to a theorem on the area-measure of a circle.



The perimeter of a regular inscribed n-gon whose side measures e_n is p_n where $p_n = n \cdot e_n$. If the apothem of the n-gon is a_n, then the area-measure of the n-gon is $\frac{1}{2} a_n p_n$ [Theorem 11-3]. If you consider

a sequence of inscribed regular polygons with, successively, more and more sides, then the perimeters of these polygons approximate more and more closely the circumference, $2\pi r$, of the circle. Also, the measure of an apothem approximates more and more closely the radius, r, of the circle. Hence, the area-measures of the polygons approximate more and more closely $\frac{1}{2}r(2\pi r)$, or πr^2.



$$c = 2\pi r$$

$$K(\bullet) = \pi r^2$$

So, we have the following theorem:

> <u>Theorem 11-4.</u>
> The area-measure of a circular region is the product of π and the square of the radius of the boundary.

* * *

<u>B.</u> 1. Find the circumference of a circle whose radius is

(a) 12 (b) $\frac{3}{\pi}$ (c) $4\sqrt{2}$ (d) 25.92

2. Find the area-measure of a circle whose radius is

(a) 6 (b) $\frac{9}{\pi}$ (c) $8\sqrt{3}$ (d) 40.2

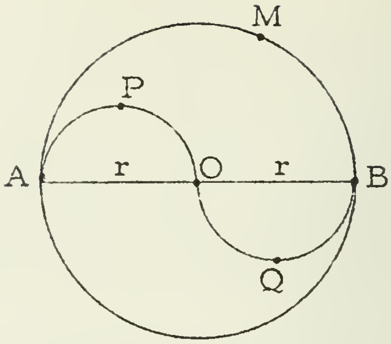
3. Find the radius of a circle whose area-measure is

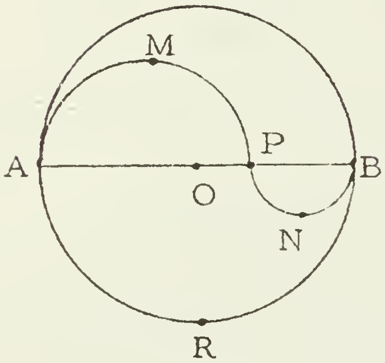
(a) 64π (b) 8π (c) 100 (d) 39.69π

4. Find the radius of a circle whose area-measure is equal to the area-measure of a square whose side measures 13.

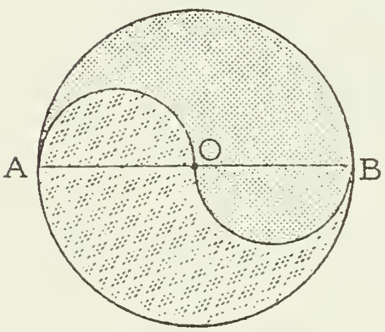
5. A circular ring is the region between two concentric circles. Find the area-measure of a circular ring whose concentric circles have radii 5 and 13, respectively.

C. 1. Prove that the sum of the measures of semicircles $\overset{\frown}{APO}$ and $\overset{\frown}{OQB}$ is the measure of semicircle $\overset{\frown}{AMB}$.



2. 

Prove that if $\overleftrightarrow{AB}$ is a diameter then the sum of the measures of semicircles $\overset{\frown}{AMP}$ and $\overset{\frown}{PNB}$ is the measure of semicircle $\overset{\frown}{ARB}$.

3. 

Prove that if $\overleftrightarrow{AB}$ is a diameter, the shaded regions have the same measure.

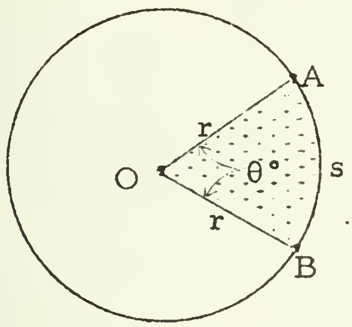
4. Prove that ratio of the circumferences of two circles is the ratio of their radii, and the ratio of the area-measures is the square of the ratio of their radii. [Draw an analogy between pairs of similar polygons and pairs of circles. It is customary to say that each two circles are similar.]

5. Prove that the area-measures of two circles are proportional to the squares of their diameters.

6. The area-measure of a certain circle is 144. If its radius is 3 times the radius of a smaller circle, find the area-measure of the smaller circle.

AREA-MEASURE FOR CIRCULAR SECTORS

The same argument which leads to the formula for the area-measure of a circle can be used in arriving at a formula for the area-measure of the region bounded by a <u>circular</u> <u>sector</u>.



(1) $K(\text{⌔}AOB) = \frac{1}{2}rs$

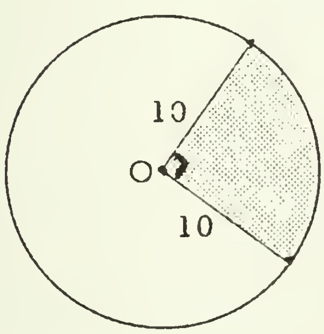
(2) If θ is the measure of $\angle AOB$,

$$K(\text{⌔}AOB) = \frac{\theta}{360} \cdot \pi r^2.$$

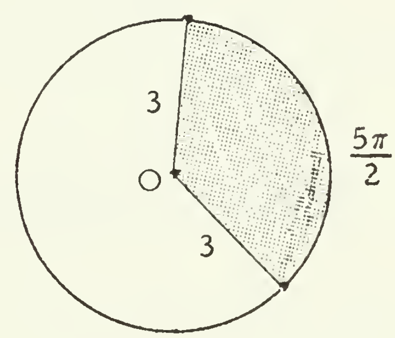
EXERCISES

<u>A.</u> Find the measures of the shaded regions.

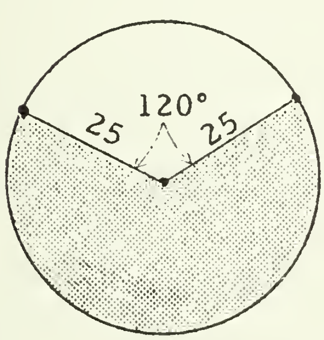
1.



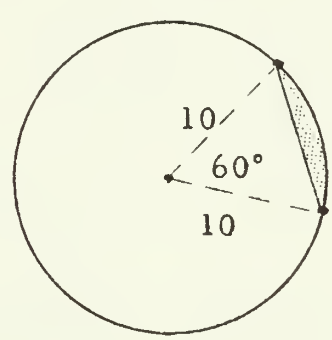
2.



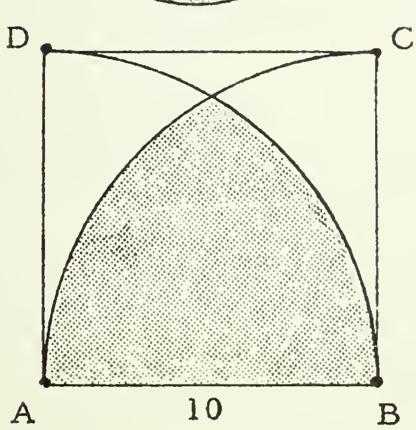
3.



4.



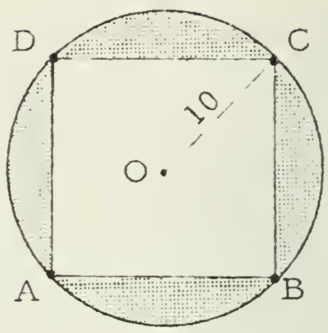
5.



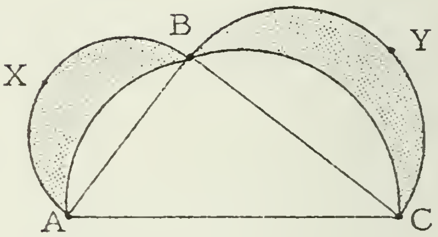
ABCD is a square each of whose sides measures 10. $\overset{\frown}{AC}$ and $\overset{\frown}{BD}$ are arcs of circles with centers A and B, respectively.

6.

7.

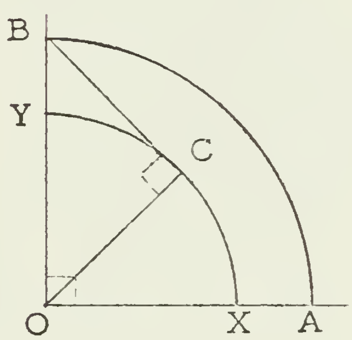


8. $\overset{\frown}{AXB}$, $\overset{\frown}{BYC}$, and $\overset{\frown}{ABC}$ are semicircles. Prove that the sum of the measures of the shaded regions is equal to the area-measure of $\triangle ABC$.



B. Solve the following problems.

1.



Hypothesis: $\angle AOB$ is a right angle, $\triangle OCB$ is an isosceles right triangle

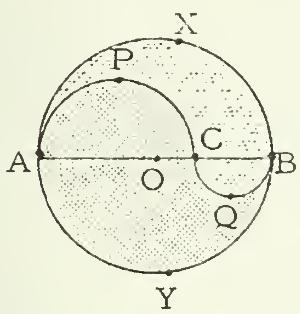
Conclusion: area-measure of sector OXCY is half the area-measure of sector OAB

2. A goat is tethered at one corner of a barn by a 75-foot rope. The barn is 60 feet by 75 feet. How many square feet of ground can the goat graze in?

3. How long is the belt which connects two pulleys each of radius 6 inches if the centers of the pulleys are 30 inches apart?

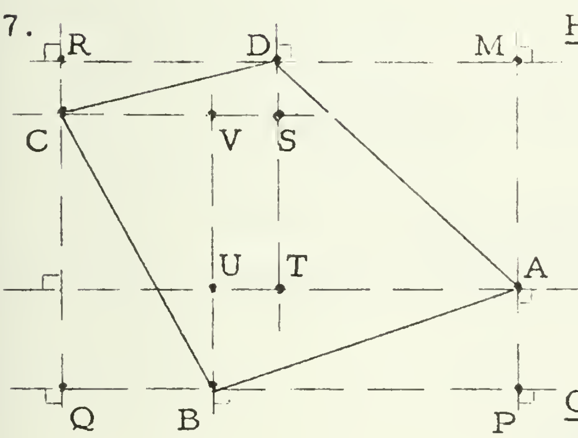
4. Three barrels are to be strapped together with a metal band. If the barrels are each 2.5 feet in diameter, how long a band is needed so that each barrel touches the others. [Disregard the amount of band needed for overlapping in fastening.]

5. The weight of ten feet of copper wire which is 1/4-inch in diameter is how many times the weight of ten feet of copper wire 1/16-inch in diameter?

6. 

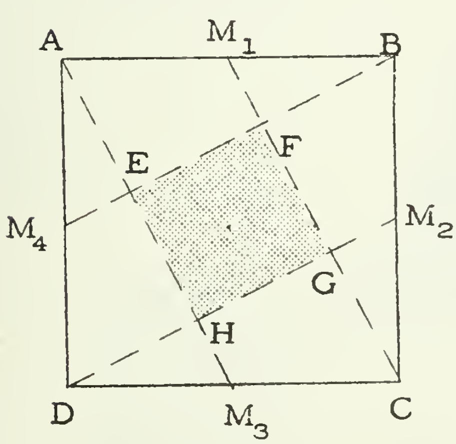
<u>Hypothesis:</u> $AC = k \cdot CB$, $\overleftrightarrow{AB}$ is a diameter

<u>Conclusion:</u> area-measure of APCQBY is k times area-measure of APCQBX

7. 

<u>Hypothesis:</u> $\overleftrightarrow{MR}$, $\overleftrightarrow{CS}$, $\overleftrightarrow{AU}$, and $\overleftrightarrow{PQ}$ are parallel, $\overleftrightarrow{MP}$, $\overleftrightarrow{DS}$, $\overleftrightarrow{VB}$, and $\overleftrightarrow{RQ}$ are parallel, $\overleftrightarrow{MP} \perp \overleftrightarrow{PQ}$

<u>Conclusion:</u> area-measure of ABCD is half the sum of the area-measure of STUV and the area-measure of MPQR

8. 

<u>Hypothesis:</u> ABCD is a square, M_1, M_2, M_3, and M_4 are midpoints

<u>Conclusion:</u> EFGH is a square

☆If k is the area-measure of ABCD, what part of k is the area-measure of EFGH?

SUMMARY OF SECTION 6.11

Notation and terminology

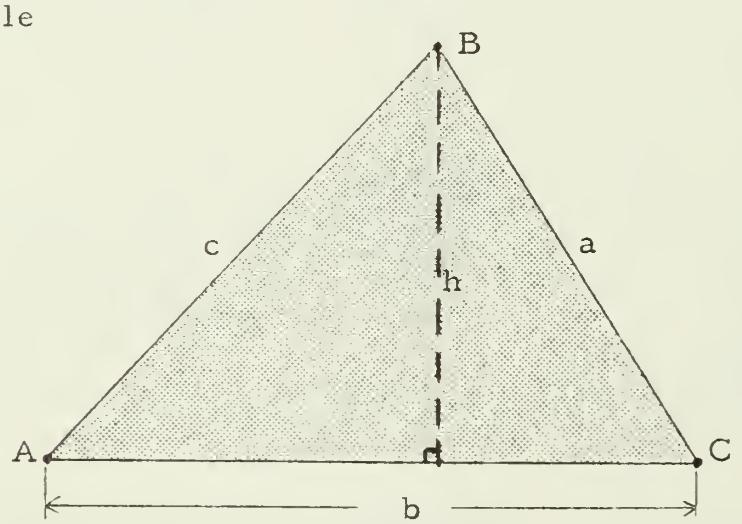
Axioms

Axiom I. The area-measure of a triangular region is half the product of the measure of any of its sides by the measure of the altitude from the opposite vertex of the triangle.

Axiom J. The area-measure of a polygonal region is the sum of the area-measures of any set of component triangular regions into which it can be cut.

Formulas

Triangle



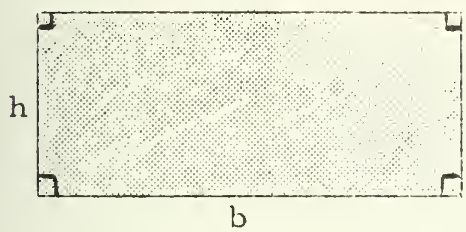
K(▲ABC) $= \frac{1}{2}hb$

K(▲ABC) $= \frac{1}{2}ab \sin \angle C$

If $s = \dfrac{a+b+c}{2}$ then K(▲ABC) $= \sqrt{s(s-a)(s-b)(s-c)}$.

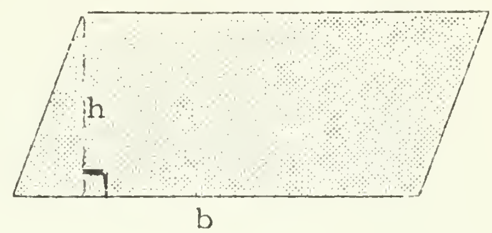
Rectangle

$K = hb$



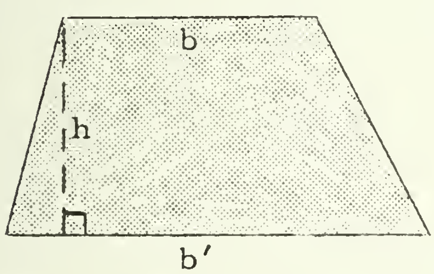
Parallelogram

$K = hb$



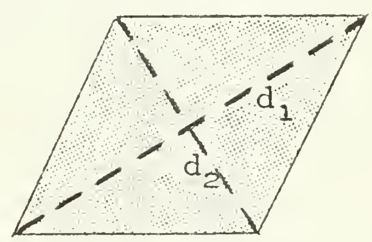
Trapezoid

$K = \frac{1}{2}h(b + b')$



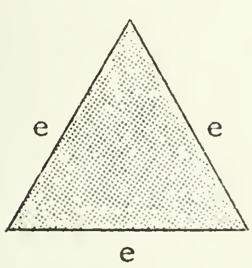
Rhombus

$K = \frac{1}{2} \cdot d_1 d_2$



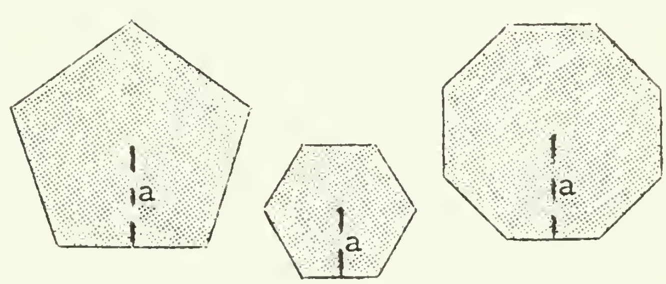
Equilateral Triangle

$K = \frac{e^2\sqrt{3}}{4}$



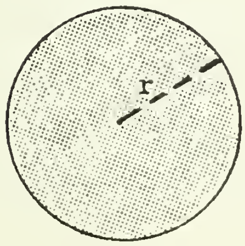
Regular Polygon

$K = \frac{1}{2}ap$



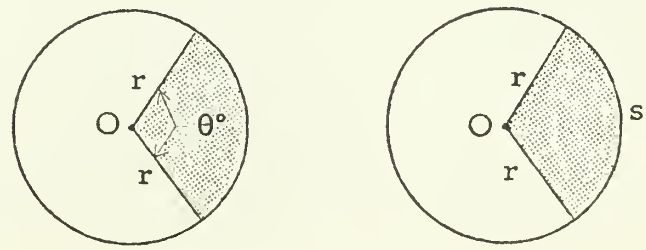
Circle

$K = \pi r^2$



Circular Sector

$K = \frac{\theta}{360} \cdot \pi r^2$ or $K = \frac{1}{2}rs$



Theorems

11-1. The area-measure of a rectangular region is equal to the product of the measures of a pair of adjacent sides of the boundary of the region.

11-2. The area-measure of the union of two polygonal regions whose intersection is contained in the boundary of each of them is the sum of the area-measures of the given polygonal regions.

11-3. The area-measure of a regular polygonal region is half the product of the apothem and the perimeter of the boundary.

11-4. The area-measure of a circular region is the product of π and the square of the radius of the boundary.

APPENDIX

The rules of reasoning. --In this section we shall discuss some of the rules of reasoning and principles of logic used in proving theorems. We shall first illustrate their use in proofs of algebra theorems which are already familiar to you. Then we shall call attention to some of the places where they are used in proving geometry theorems.

Don't expect to gain complete control over these rules on the first reading of this section. The ability to use them will grow as you think about applying them in writing proofs and in reading proofs. You will probably find it helpful to reread this section at various times while you are studying this unit.

UNIVERSAL INSTANTIATION and THE SUBSTITUTION RULE FOR EQUATIONS

Let's start with a proof of the kind you may have written back in Unit 2.

Prove: $\forall_x\ x(x + 1) = xx + x$

What you did in Unit 2 was to write a test-pattern, a pattern which shows how you would verify each instance of the universal generalization.

$$\begin{aligned} a(a + 1) &= aa + a1 && [\forall_x \forall_y \forall_z\ x(y + z) = xy + xz] \\ &= aa + a && [\forall_x\ x1 = x] \end{aligned}$$

So, $a(a + 1) = aa + a$.

The test-pattern justifies our saying that '$\forall_x\ x(x + 1) = xx + x$' is a theorem, because it shows how any instance of the generalization can be derived from a previously proved theorem and a basic principle. [What theorem, and what basic principle?]

To see more clearly the reasoning involved in the proof, it will be helpful to rewrite it in an expanded form. In trying to prove the generalization, your first thought might have been to take the expression '$a(a + 1)$' and transform it. There are many ways to do this. But, perhaps you thought of using the left distributive principle for multiplication over addition. Doing so produces the sentence '$a(a + 1) = aa + a1$'. This sentence follows from the ℓdpma because it is an instance of it.

Now, you look at the new expression 'aa + a1'. You are trying to get to 'aa + a'. You think of applying the principle for multiplying by 1: $\forall_x\ x1 = x$. An <u>instance</u> of this generalization is 'a1 = a'. So, you take the sentence:

$$a(a + 1) = aa + a1,$$

and <u>substitute</u> the right side, 'a', of the equation 'a1 = a' for the 'a1'. This produces the new sentence:

$$a(a + 1) = aa + a$$

This is an instance of the generalization you are trying to prove. Since your reasoning would hold for any instance, you conclude that the generalization itself has been proved.

Now, let's put these steps together into what is called <u>a column proof</u>.

(1)	$\forall_x \forall_y \forall_z\ x(y + z) = xy + xz$	[theorem]
(2)	$a(a + 1) = aa + a1$	[instance of (1)]
(3)	$\forall_x\ x1 = x$	[basic principle]
(4)	$a1 = a$	[instance of (3)]
(5)	$a(a + 1) = aa + a$	[substituting from (4) into (2)]
(6)	$\forall_x\ x(x + 1) = xx + x$	[(1) - (5) form a test-pattern]

The column of sentences is a proof, the last sentence being the theorem which was to be proved. Each of the bracketed comments on the right explains the corresponding step.

For example, since we are trying to show that (6) is a theorem, it is permissible to use basic principles and previously proved theorems as steps in the proof. The comments tell that step (1) is a previously proved theorem, and that step (3) is a basic principle.

The comments for steps (2) and (4) indicate that each of these is an instance of a previous step. So, each of these steps is justified by a rule of reasoning called <u>universal instantiation.</u>

<u>Universal instantiation.</u>
Each instance of a universal generalization is a consequence of it.

Another rule of reasoning--<u>the substitution rule for equations</u>--justifies step (5). This rule says that step (5) is a consequence of steps (2) and (4) because (4) is an equation and (5) is obtained by putting the right side of (4) in place of its left side somewhere in (2). Schematically:

$$\frac{(4)\quad a1 = \boxed{a} \qquad (2)\quad a(a+1) = aa + a1}{(5)\quad a(a+1) = aa + \boxed{a}}$$

[This way of writing steps (2), (4), and (5)--(2) and (4) above a horizontal bar and (5) directly below it--is a fast way of saying that (5) is a consequence of steps (2) and (4). Since the substitution is made from (4) into (2), it is customary to write (4) first.]

The substitution rule for equations.
Given an equation and another sentence, if the left side of the equation is replaced by its right side somewhere in the sentence, the new sentence thus obtained is a consequence of the given equation and sentence.

Use the substitution rule to obtain a consequence of the equation '$x = 2$' and the sentence '$5 + x > 1$'.

Returning now to the column proof, we see that the comment for step (6) tells us that steps (1)-(5) form a test-pattern for statements of the form of (5), that is, for instances of (6). So, the universal generalization of (5)--that is, step (6)--is a consequence of the theorem (1) and the basic principle (3). So, since (6) is a consequence of theorems and basic principles, (6) is a theorem.

Another way of seeing the logical connections among the steps of the column proof is to make a diagram of its structure.

For example, to show that (2) is a consequence of (1) we can write:

$$\frac{(1)}{(2)}$$

[You can read this aloud as: (1), therefore (2). It tells you that

the second step of the proof is a consequence of, or follows from, the first step, that is, that the first step implies the second.]

Alongside of this, we indicate that (4) follows from (3):

$$\frac{(3)}{(4)} \qquad \frac{(1)}{(2)}$$

Then, to show that (5) follows from (4) and (2), we add to the diagram to get:

$$\dfrac{\dfrac{(3)}{(4)} \qquad \dfrac{(1)}{(2)}}{(5)}$$

Finally, we show that steps (1) - (5) form a test-pattern for (6) by writing a double-bar below the '(5)' and writing a '(6)' below the double-bar:

$$\begin{array}{c} \dfrac{\dfrac{(3)}{(4)} \qquad \dfrac{(1)}{(2)}}{(5)} \\ \hline\hline (6) \end{array}$$

This diagram shows that (2) and (4) are logical consequences of (1) and (3), and that (5) is a logical consequence of (2) and (4). So, it shows that (5) is a logical consequence of (1) and (3). Finally, it shows that (6) is a logical consequence of (1) and (3).

EXERCISES

<u>A.</u> Each of the following exercises refers to an example of inferring a <u>conclusion</u> from a <u>premiss</u> by means of the logical principle of universal instantiation. Your job is to write in a sentence which completes the pattern.

<u>Sample 1.</u> $\dfrac{\forall_X \forall_Y \forall_Z \text{ if } Y \in \overline{XZ} \text{ then } Y \in \overleftrightarrow{XZ}}{\qquad}$

<u>Solution.</u> $\dfrac{\forall_X \forall_Y \forall_Z \text{ if } Y \in \overline{XZ} \text{ then } Y \in \overleftrightarrow{XZ}}{\text{if } B \in \overline{AC} \text{ then } B \in \overleftrightarrow{AC}}$

Sample 2.

$$\frac{\qquad\qquad}{a^2 - b^2 = (a - b)(a + b)}$$

Solution. $$\frac{\forall_x \forall_y\, x^2 - y^2 = (x - y)(x + y)}{a^2 - b^2 = (a - b)(a + b)}$$

1. $\forall_X \forall_Y \forall_Z \forall_W$ if $Y \in \overline{XZ}$ and $Z \in \overline{YW}$ then $Z \in \overline{XW}$
____

2. $\forall_x \forall_y\ (x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
____

3. ______________________________
$(5 - 4)^3 = 5^3 - 3 \cdot 5^2 \cdot 4 + 3 \cdot 5 \cdot 4^2 - 4^3$

4. ______________________________
if $a + b = c$ then $a = c - b$

B. Each of the following exercises refers to an example of inferring a conclusion from two premisses by means of the substitution rule for equations. Your job is to find a sentence which completes the pattern.

Sample. $8 = 6 + 2 \qquad 7 \cdot 8 > 5 \cdot 8$
____

Solution. The left side of the equation is '8'. We can obtain a conclusion by putting the right side, '6 + 2', of the equation in place of one or both of the '8's in the second premiss.

$$\frac{8 = 6 + 2 \qquad 7 \cdot 8 > 5 \cdot 8}{7 \cdot (6 + 2) > 5 \cdot 8}$$

There are two other possible conclusions. What are they?

1. $3 = 2 + 1 \qquad 8 \cdot 3 > 23$
____

2. $$\frac{3 = 2 + 1}{8 \cdot 3 > 7 \cdot (2 + 1)}$$

3. $$\frac{a = b + c \qquad a^2 - d^2 = (a - d)(a + d)}{\qquad}$$

4. $$\frac{\overleftrightarrow{AB} = \overleftrightarrow{BA} \qquad \text{the length of } \overleftrightarrow{AB} = \text{the length of } \overleftrightarrow{AB}}{\qquad}$$

5. $$\frac{b = d}{\text{if } a > d \text{ and } d > c \text{ then } a > c}$$

6. $$\frac{a = b}{b = a}$$

*

Notice that in Exercise 6, if you wrote 'a = a' as the second premiss, you then inferred 'b = a' from the premisses 'a = b' and 'a = a'. Now, 'a = a' is an instance of the logical principle of identity [''a thing is the same as itself'']. This indicates a procedure which you can use to ''turn an equation around''.

For example, suppose you start with the two premisses:

$$(1)\ a = b \qquad\qquad (2)\ b^2 > 0,$$

and your job is to derive the conclusion: (3) $a^2 > 0$. According to our substitution rule, you can replace any 'a' in (2) by a 'b'. But, the rule does not say you can replace any occurrence of the right side of (1) in (2) by the left side of (1). We can take care of this by introducing the premiss 'a = a':

$$\dfrac{\dfrac{(1)\ a = b \qquad a = a}{b = a} \qquad (2)\ b^2 > 0}{(3)\ a^2 > 0}$$

So, by means of two applications of the substitution rule we can derive (3) from (1) and (2) together with the logical principle of identity.

Since we can always turn equations around this way by bringing in the principle of identity, we shall feel free from now on to replace either side of an equation by the other when we use the substitution rule.

*

7. $$\frac{2 = a \qquad 3a + 7 = 13}{}$$

8. $$\frac{b = a \qquad\qquad\qquad}{b = c}$$

9. $$\frac{\qquad\qquad 4a + 3a = 7a}{4 \cdot 5 + 3 \cdot 5 = 7a}$$

10. $$\frac{a + 0 = a \qquad a(a + 0) = aa + a0}{}$$

11. $$\frac{a = b \qquad b = b}{}$$

12. $$\frac{a = b \qquad b = a}{}$$

13. $$\frac{a = b \qquad\qquad}{ac = bc}$$

14. $$\frac{c = 0 \qquad a + b > c}{}$$

15. $$\frac{\qquad\qquad \text{if } a = b \text{ then } a + c = b + c}{\text{if } a = b \text{ then } a + d = b + d}$$

16. $$\frac{A = B \qquad \text{if } A \in \overline{BC} \text{ then } \overleftrightarrow{BC} \cap \ell = \{A\}}{}$$

17. $$\frac{\text{Sacramento is the capital of California} \qquad \text{James lives in Sacramento}}{}$$

18. $$\frac{\qquad\qquad \text{M is the midpoint of } \overline{AB}}{\text{N is the midpoint of } \overline{AB}}$$

19. $$\frac{\text{the midpoint of } \overline{AB} \text{ is the midpoint of } \overline{CD}}{\text{M is the midpoint of } \overline{CD}}$$

C. Here is a test-pattern for the theorem:

$$\forall_u \forall_v \forall_x \forall_y\, (uv)(xy) = (ux)(vy)$$

$$
\begin{aligned}
(ab)(cd) &= a[b(cd)] && [\forall_x \forall_y \forall_z\, (xy)z = x(yz)] \\
&= a[(cd)b] && [\forall_x \forall_y\, xy = yx] \\
&= a[c(db)] && [\forall_x \forall_y \forall_z\, (xy)z = x(yz)] \\
&= a[c(bd)] && [\forall_x \forall_y\, xy = yx] \\
&= (ac)(bd) && [\forall_x \forall_y \forall_z\, (xy)z = x(yz)]
\end{aligned}
$$

So, $(ab)(cd) = (ac)(bd)$.

1. Now, here is a column proof for the same theorem. Your job is to write the marginal comments which justify the steps.

(1) $\forall_x \forall_y \forall_z\ (xy)z = x(yz)$ [____________________]

(2) $(ab)(cd) = a[b(cd)]$ [____________________]

(3) $\forall_x \forall_y\ xy = yx$ [____________________]

(4) $b(cd) = (cd)b$ [____________________]

(5) $(ab)(cd) = a[(cd)b]$ [____________________]

(6) $(cd)b = c(db)$ [____________________]

(7) $(ab)(cd) = a[c(db)]$ [____________________]

(8) $db = bd$ [____________________]

(9) $(ab)(cd) = a[c(bd)]$ [____________________]

(10) $(ac)(bd) = a[c(bd)]$ [____________________]

(11) $(ab)(cd) = (ac)(bd)$ [____________________]

(12) $\forall_u \forall_v \forall_x \forall_y\ (uv)(xy) = (ux)(vy)$ [____________________]

2. Here is a diagram which shows the logical connections among the first six steps of this proof. Finish the diagram for the entire proof.

$$
\begin{array}{c}
\dfrac{(1)}{(6)} \qquad \dfrac{\dfrac{(3)}{(4)} \quad \dfrac{(1)}{(2)}}{(5)} \\
\hline
\end{array}
$$

EXPLORATION EXERCISES

Each of the following exercises contains two premisses and a conclusion. Your job is to decide in each case if the conclusion follows logically from the premisses. [Note that you are not being asked to decide whether the premisses or conclusion are true.]

1. $$\frac{2 = 2 \qquad \text{if } 2 = 2 \text{ then } 2 + 3 = 2 + 3}{2 + 3 = 2 + 3}$$

2. $$\frac{a = b \qquad \text{if } a = b \text{ then } a + c = b + c}{a + c = b + c}$$

3. $$\frac{\text{bats are birds} \qquad \text{if bats are birds then bats lay eggs}}{\text{bats lay eggs}}$$

4. $$\frac{\text{Ed lives in Iowa} \qquad \text{if Ed lives in Ames then Ed lives in Iowa}}{\text{Ed lives in Ames}}$$

5. $$\frac{ac = bc \qquad \text{if } a = b \text{ then } ac = bc}{a = b}$$

6. $$\frac{A \in \overline{BC} \qquad \text{if } A \in \overline{BC} \text{ then } m(\overline{BA}) + m(\overline{AC}) = m(\overline{BC})}{m(\overline{BA}) + m(\overline{AC}) = m(\overline{BC})}$$

7. $$\frac{a = 1 \qquad \text{if } a + 1 = 2 \text{ then } a = 1}{a + 1 = 2}$$

8. $$\frac{2 = 3 \qquad \text{if } 2 = 3 \text{ then } 2 + 5 = 3 + 5}{2 + 5 = 3 + 5}$$

9. $$\frac{7 = 8 \qquad \text{if } 2 = 3 \text{ then } 7 = 8}{2 = 3}$$

10. $$\frac{a \neq b \qquad \text{if } a \neq b \text{ then } a > b \text{ or } a < b}{a > b \text{ or } a < b}$$

11. $$\frac{\boxed{} \qquad \text{if } \boxed{} \text{ then } \bigcirc}{\bigcirc}$$

12. $$\frac{\bigcirc \qquad \text{if } \boxed{} \text{ then } \bigcirc}{\boxed{}}$$

MODUS PONENS

In the preceding exercises you used a rule of reasoning which enables you to infer a conclusion from two premisses, one of which is an if-then sentence. You used this rule of reasoning many times in proving algebra theorems. For example, here is a column proof of the familiar algebra theorem:

$$\forall_x \; --x = x$$

[You may recall that a proof of this theorem uses the 0-sum theorem. Notice how the new rule of reasoning is used.]

(1)	$\forall_x \; x + -x = 0$	[basic principle]
(2)	$a + -a = 0$	[(1)]
(3)	$\forall_x \forall_y \; x + y = y + x$	[basic principle]
(4)	$a + -a = -a + a$	[(3)]
(5)	$-a + a = 0$	[(2) and (4)]
(6)	$\forall_x \forall_y$ if $x + y = 0$ then $-x = y$	[theorem]
(7)	if $-a + a = 0$ then $--a = a$	[(6)]
(8)	$--a = a$	[by modus ponens from (5) and (7)]
(9)	$\forall_x \; --x = x$	[(1) - (8)]

Here is a diagram of the structure of this proof:

$$\begin{array}{c} \dfrac{\dfrac{\dfrac{(3)}{(4)} \qquad \dfrac{(1)}{(2)}}{(5)} \qquad \dfrac{(6)}{(7)}}{(8)} \\ \hline\hline (9) \end{array}$$

Sentence (9) is a theorem because it is a consequence of the basic principles (1) and (3), and the previously proved theorem (6).

Looking at the marginal comments [or the diagram], we see that (2) is inferred from (1). What rule of reasoning justifies this inference?

There are two more applications of this rule in the proof. Tell where.

What rule justifies inferring (5) from (2) and (4)?

Step (8) follows from (5) and the if-then sentence (7) by the new rule of reasoning called <u>modus ponens</u>. This rule says that from an if-then sentence:

$$(7)\quad \text{if } \underbrace{-a + a = 0}_{\text{antecedent}} \qquad \text{then } \underbrace{--a = a}_{\text{consequent}}$$

and its antecedent:

$$(5)\quad -a + a = 0,$$

you may infer its consequent:

$$(8)\quad --a = 0$$

Schematically, modus ponens can be expressed by:

$$\frac{\square \qquad \text{if } \square \text{ then } \bigcirc}{\bigcirc}$$

More briefly, using letters instead of frames:

<u>Modus ponens.</u>

$$\frac{p \qquad \text{if } p \text{ then } q}{q}$$

To get an example of the use of modus ponens, all you do is replace 'p' and 'q' by sentences. The sentence you put in place of 'p' is the antecedent of the if-then sentence, and the sentence you put in place of 'q' is the consequent of the if-then sentence.

Turn back to the Exploration Exercises on page 6-365. For each exercise which illustrates modus ponens, show that it does so by telling what sentences the letters 'p' and 'q' must be replaced by in order to obtain the exercise. [<u>Sample.</u> Exercise 1 illustrates modus ponens because it can be obtained from the scheme displayed in the box above by replacing 'p' by '2 = 2' and 'q' by '2 + 3 = 2 + 3'.]

Explain, in terms of replacing 'p' and 'q' by sentences, why Exercise 4 does not illustrate modus ponens.

A FALLACY

Each inference which fits the scheme of Exercise 11 on page 6-365 [modus ponens] is <u>valid</u>--that is, its conclusion follows logically from its premisses.

Compare the scheme in Exercise 12 with that of Exercise 11. Exercise 4 is an inference which fits the scheme of Exercise 12. However, as you have probably already decided, Exercise 4 is not a valid inference. [Although Ames is in Iowa, so the second premiss is true, Ed may live in Iowa without living in Ames.]

One who makes the error of thinking that the antecedent of an if-then sentence follows from the sentence itself and its <u>consequent</u> is said to have committed the <u>fallacy</u> of "affirming the consequent". Other examples of this fallacy occur in the following arguments:

> For each x, if $x = -1$ then $x^2 = 1$. So, since 1 is a root of '$x^2 = 1$', 1 is also a root of '$x = -1$'.
>
> For each x, if $x + 3 = 4$ then $x = 1$. So, since 1 is a root of '$x = 1$', 1 is also a root of '$x + 3 = 4$'.

EXERCISES

<u>A.</u> Each of the following exercises refers to an example of inferring a conclusion from two premisses by means of modus ponens. Your job is to give the missing sentence. Also, for each exercise, tell which sentence is the antecedent of the if-then sentence and which sentence is its consequent.

1. $a = b$ if $a = b$ then $c = d$
   ____

2. ______   if $A \in \overline{BC}$ then $A \in \overleftrightarrow{BC}$
   ____
   $A \in \overleftrightarrow{BC}$

3. $a + b = 0$ ______
   ____
   $b = -a$

4. ______   if $ab = c$ then $b = c \div a$
   ____
   $b = c \div a$

5. $a - b \neq 0$ ______
   ____
   $a \neq b$

6. $A \in \ell$ if $A \in \ell$ then $A \in m$
   ____

7. $A \notin \ell$ ______
   ____
   $A \notin m$

8. $\ell \parallel m$ ______
   ____
   $\ell \cap m = \emptyset$

9.
$$\dfrac{\text{Bill lives in Dallas} \qquad \begin{array}{l}\text{if Bill lives in Dallas}\\ \text{then Bill lives in Texas}\end{array}}{}$$

10.
$$\dfrac{\text{Bill lives in Texas}}{\text{Bill lives in Dallas}}$$

11.
$$\dfrac{\text{Bill does not live in Texas}}{\text{Bill does not live in Dallas}}$$

B. Each of the following exercises gives a pattern for a sequence of inferences. Write sentences which complete the pattern, and be prepared to tell the rule of reasoning which justifies each inference.

Sample.

$$\dfrac{d = e \qquad \dfrac{a \neq b \qquad \text{if } a \neq b \text{ then } c \neq d}{}}{}$$

Solution. From the first two premisses we can infer '$c \neq d$' by modus ponens. Then, from '$d = e$' and '$c \neq d$' we can, by the substitution rule, infer '$c \neq e$'. Hence, the completed pattern is:

$$\dfrac{d = e \qquad \dfrac{a \neq b \qquad \text{if } a \neq b \text{ then } c \neq d}{c \neq d}}{c \neq e}$$

1.
$$\dfrac{\dfrac{a = b \qquad a \neq c}{} \qquad }{b \neq d}$$

2.
$$\dfrac{\dfrac{a = b}{} \qquad b = a}{a = c}$$

3.
$$\dfrac{\ell = n \qquad \dfrac{ \qquad \text{if } \ell \parallel m \text{ then } m \parallel \ell}{m \parallel \ell}}{}$$

4.
$$\dfrac{C \in \overline{BC} \qquad \dfrac{ \qquad \text{if } A \in \overline{BC} \text{ then } \overleftrightarrow{BC} \cap \ell = \{A\}}{}}{\overleftrightarrow{BC} \cap \ell = \{A\}}$$

C. 1. Mr. Jones is checking attendance. He notices that Margaret is absent. He knows that if Margaret were ill, she would be absent. So, he concludes that Margaret must be ill. Is his reasoning valid?

2. Steve knows that if two angles are right angles then they have the same number of degrees. Steve's teacher asks him to draw pictures of two angles with the same number of degrees. He concludes that his teacher wants him to draw two right angles, and he does so. Is Steve's reasoning correct? Did Steve carry out the teacher's instructions?

D. Here is a proof of the algebra theorem '$\forall_x$ x0 = 0'.

Since a + 0 = a, it follows from the uniqueness principle for multiplication that (a + 0)a = aa. But, (a + 0)a = aa + 0a. So, aa + 0a = aa. But, aa + 0 = aa, also. So, aa + 0a = aa + 0 and, by the cancellation principle for addition, 0a = 0. Since a0 = 0a, it follows that a0 = 0. Consequently, for each x, x0 = 0.

1. Here is an incomplete column proof using the same reasoning. Fill in the blanks and the missing marginal comments.

(1)	$\forall_x$ x + 0 = x	[____________________]
(2)	a + 0 = a	[____________________]
(3)	____________________	[theorem]
(4)	if a + 0 = a then (a + 0)a = aa	[(3)]
(5)	(a + 0)a = aa	[by modus ponens from (2) and (4)]
(6)	____________________	[basic principle]
(7)	(a + 0)a = aa + 0a	[____________________]
(8)	aa + 0a = aa	[____________________]
(9)	aa + 0 = aa	[(1)]
(10)	____________________	[(8) and (9)]

(11)	____________________	[theorem]
(12)	____________________	[(11)]
(13)	$0a = 0$	[____________________]
(14)	____________________	[basic principle]
(15)	$a0 = 0a$	[(14)]
(16)	$a0 = 0$	[____________________]
(17)	$\forall_x \; x0 = 0$	[(1) - (16)]

2. Make a diagram showing the structure of the proof.

<u>E.</u> The algebra theorem of Part D can be used to prove another theorem:

(*) $\forall_x \forall_y$ if $y = 0$ then $xy = 0$

Here is a paragraph proof of (*):

Suppose that $b = 0$. It follows that $ab = a0$. So, since $\forall_x \; x0 = 0$, it follows that $ab = 0$. Hence, <u>if</u> $b = 0$ then $ab = 0$. Consequently, $\forall_x \forall_y$ if $y = 0$ then $xy = 0$.

1. By what theorem does it follow that $ab = a0$?

2. What theorems is (*) a consequence of?

CONDITIONALIZING, and DISCHARGING AN ASSUMPTION.

You have seen how to draw a conclusion from a conditional sentence [that is, an if-then sentence] and its antecedent by using modus ponens. Now, let's see how to derive a conditional sentence from others.

This kind of problem is much more common in geometry than in algebra. But, you did it several times in your algebra course. For example, do you recall how you proved the uniqueness principle for addition? The theorem is:

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } x + z = y + z$$

So, your job was to derive an instance like:

$$\text{if } a = b \text{ then } a + c = b + c$$

You started by supposing the antecedent and then you derived the consequent. Here is a test-pattern:

Suppose that	$a = b.$	
Since	$a + c = a + c,$	$[\forall_x\ x = x]$
it follows that	$a + c = b + c.$	
Hence,	if $a = b$ then $a + c = b + c.$	

The first three lines of the test-pattern show that the consequent of the conditional sentence follows from the principle of identity and the assumption [or: supposition] 'a = b'. As the fourth line indicates, the conditional sentence:

$$\underline{\text{if}}\ a = b \text{ then } a + c = b + c$$

[whose antecedent is the assumption and whose consequent is the sentence in the third line] is a consequence of the principle of identity alone.

Let's rewrite the proof in column form:

(1)	$a = b$	[assumption]
(2)	$\forall_x\ x = x$	[logical principle]
(3)	$a + c = a + c$	[(2)]
(4)	$a + c = b + c$	[(1) and (3)]
(5)	if $a = b$ then $a + c = b + c$	[conditionalizing (4); discharge (1)]
(6)	$\forall_x \forall_y \forall_z$ if $x = y$ then $x + z = y + z$	[(1) - (5)]

Here is a diagram of the proof:

$$\begin{array}{c} \overset{*}{(1)} \qquad \dfrac{(2)}{(3)} \\ \hline \dfrac{(4)}{(5)}\;{*} \\ \hline\hline (6) \end{array}$$

The marginal comment for step (5) refers to two new rules of reasoning. The first of these--conditionalizing--says that a conditional sentence may be inferred from its consequent. In this case:

$$\dfrac{(4)\ \ a + c = b + c}{(5)\ \ \text{if } a = b \text{ then } a + c = b + c}$$

So, since (4) follows from (1) and (2), so does (5).

Now, since (5) follows from (1) and (2), and since the antecedent of (5) is (1), the second new rule tells us that (5) follows from (2), alone. [In the diagram we use '*'s to show the application of this rule.]

Here is a scheme which suggests these two rules:

$$\dfrac{\begin{array}{c}[\,p\,] \\ q\end{array}}{\text{if } p \text{ then } q}$$

The '[p]' is to remind you of the second rule: if you derive the consequent of a conditional from its antecedent, and other premisses, then the conditional itself is a consequence of these other premisses alone. When we use this rule, we call the antecedent *an assumption*, and indicate that the conditional sentence does not depend on this assumption by saying that the assumption is *discharged*.

We shall refer to these two rules as one:

Conditionalizing, and discharging an assumption.
$$\dfrac{\begin{array}{c}[\,p\,] \\ q\end{array}}{\text{if } p \text{ then } q}$$

As shown in the proof, this rule gives you a strategy to use in deriving a conditional sentence from given premisses [basic principles, for example]. Use the antecedent of the conditional as an assumption, or an extra premiss. Then, try to derive the consequent of the conditional from this assumption together with the other premisses. When you have done this, the next step is to state the conditional, and on doing so, to discharge the extra premiss.

EXERCISES

<u>A.</u> Read the following discussion.

> Al said, "I can prove that the Queen of England lives in the United States." Stan challenged him to do so. Al replied, "O.K. Let's suppose the Queen of England lives in Chicago. Now, anyone who lives in Chicago must live in Illinois. Right? ... So, the Queen of England lives in Illinois. But, anyone who lives in Illinois must also live in the United States. So, the Queen of England lives in the United States. Finished."

1. Did Al use valid reasoning in deriving the sentence:

   The Queen of England lives in the United States.

   From what premisses does the sentence follow?

2. Stan replied that he saw how Al reached the conclusion about the Queen of England living in the United States. "Your reasoning is perfect. I even agree with the two premisses about people who live in Chicago and about people who live in Illinois. But, I don't accept the premiss at the very beginning of your argument about the Queen of England living in Chicago." If Stan accepts the two premisses he mentioned, what conclusion about the Queen should he be willing to accept on the basis of Al's argument?

<u>B.</u> Here is a proof of the algebra theorem:

$$\forall_x \forall_y \text{ if } x = -y \text{ then } -x = y$$

1. Write the marginal comments which explain the steps.

2. Make a diagram of the proof.

(1) $a = -b$ [assumption]

(2) $\forall_x \; x + -x = 0$ [__________]

(3) $b + -b = 0$ [__________]

(4) $b + a = 0$ [__________]

(5) $\forall_x \forall_y \; x + y = y + x$ [__________]

(6) $b + a = a + b$ [__________]

(7) $a + b = 0$ [__________]

(8) $\forall_x \forall_y$ if $x + y = 0$ then $-x = y$ [__________]

(9) if $a + b = 0$ then $-a = b$ [__________]

(10) $-a = b$ [__________]

(11) if $a = -b$ then $-a = b$ [_____; _____]

(12) $\forall_x \forall_y$ if $x = -y$ then $-x = y$ [__________]

C. Here is an incomplete column proof of the principle of logic '$\forall_x \forall_y$ if $x = y$ then $y = x$'. This principle justifies "turning an equation around". [The principle states that the relation of equality has the property of being symmetric.]

1. Complete the proof and write the missing marginal comments.

2. Make a diagram of the proof.

(1) __________ [assumption]

(2) __________ [logical principle]

(3) $a = a$ [__________]

(4) __________ [__________]

(5) if __________ [__________]

(6) $\forall_x \forall_y$ if __________ [__________]

☆ D. State a theorem which could be called <u>the uniqueness principle for squaring</u>. Write a column proof of the theorem.

E. Each of the following exercises shows a sequence of inferences in which the last step is obtained by conditionalizing, and discharging an assumption. Write sentences to complete the patterns.

<u>Sample.</u>

$$\frac{a \overset{*}{=} b \qquad a - c = a - c}{a - c = b - c}\;{*}$$

<u>Solution.</u> The second asterisk tells you that the next step is a conditional sentence whose consequent is the preceding step. The first asterisk tells you that the antecedent of the conditional is 'a = b'. So, the missing sentence is:

$$\text{if } a = b \text{ then } a - c = b - c$$

The pattern, when completed, shows you that the conditional sentence follows from 'a - c = a - c', alone.

1. $$\frac{a \overset{*}{=} 2 \qquad a^2 + 3a - 6 = 0}{2^2 + 3 \cdot 2 - 6 = 0}\;{*}$$

2. $$\frac{c \overset{*}{=} d \qquad \dfrac{a = b \qquad a \neq c}{b \neq c}}{b \neq d}\;{*}$$

3. $$\frac{c = d \qquad \dfrac{a = b \qquad a \overset{*}{\neq} c}{b \neq c}}{b \neq d}\;{*}$$

4. $$\frac{\;ac = ac}{\underline{\hspace{5cm}}\;{*}}$$

if ______ then $ac = bc$

5. $$\dfrac{\dfrac{\text{Ed lives in Miami} \qquad \text{if Ed lives in Miami then Ed lives in Florida}}{\qquad\qquad\qquad} \qquad \text{if Ed lives in Florida then Ed lives in the U.S.}}{\qquad\qquad\qquad\qquad \; *}$$

6. $$\dfrac{\dfrac{\overset{*}{p} \qquad \text{if } p \text{ then } q}{\qquad\qquad} \qquad \text{if } q \text{ then } r}{\qquad\qquad \; *}$$

7. $$\dfrac{a \overset{*}{=} b \qquad \text{if } a = b \text{ then } c = d}{\qquad\qquad \; *}$$

MODUS TOLLENS

(1) If Bill Matsumoto lives in Honolulu then he lives in Hawaii. As a matter of fact, (2) he does not live in Hawaii. Suppose one concludes from these two premisses that (3) Bill does not live in Honolulu. Is the reasoning valid?

The rule of inference which tells you that this reasoning is valid is called <u>modus</u> <u>tollens</u>. Here is another example:

$$\dfrac{\text{if } 2 \cdot 3 = 2 \cdot 4 \text{ then } 3 = 4 \qquad \text{not } [3 = 4]}{\text{not } [2 \cdot 3 = 2 \cdot 4]}$$

[Of course, we usually abbreviate 'not [3 = 4]' to '$3 \neq 4$'. Either of these sentences is said to be the <u>denial</u> of the sentence '3 = 4'.]

Modus tollens tells you that you can infer the denial of the antecedent of a conditional sentence from the conditional sentence together with the denial of its consequent:

<u>Modus tollens.</u>
$$\dfrac{\text{if } p \text{ then } q \qquad \text{not } q}{\text{not } p}$$

[What does modus <u>ponens</u> tell you?]

EXERCISES

A. Some of the following exercises refer to inferences justified by modus tollens, and some do not. Your job is to pick out the ones that do, and in those cases, write in the missing sentences.

1. $\dfrac{\text{if } a = b \text{ then } ac = bc \qquad ac \neq bc}{}$

2. $\dfrac{\text{if } a = b \text{ then } -a = -b}{a \neq b}$

3. $\dfrac{\text{if } A \in \overline{BC} \text{ then } A \in \overset{\bullet\!-\!\bullet}{BC} \qquad A \notin \overset{\bullet\!-\!\bullet}{BC}}{}$

4. $\dfrac{\text{if } A \in \overline{BC} \text{ then } A \in \overset{\bullet\!-\!\bullet}{BC} \qquad A \notin \overline{BC}}{}$

5. $\dfrac{\text{if } A \in \overline{BC} \text{ then } A \in \overset{\bullet\!-\!\bullet}{BC} \qquad A \in \overline{BC}}{}$

6. if Tom lives in Atlanta then Tom lives in Georgia — Tom does not live in Georgia
   ____

7. if Tom lives in Atlanta then Tom lives in Georgia — Tom does not live in Atlanta
   ____

8. if it is raining then I'll eat my hat — I'll not eat my hat
   ____

* * *

In discussing modus ponens [page 6-367], we mentioned a type of invalid reasoning called <u>affirming the consequent</u>.

[modus ponens]	[affirming the consequent]
$\dfrac{p \qquad \text{if } p \text{ then } q}{q}$	$\dfrac{\text{if } p \text{ then } q \qquad q}{p}$ INVALID

An example of affirming the consequent is inferring '5 = 7' from the

conditional sentence 'if $5 = 7$ then $5 \cdot 0 = 7 \cdot 0$' and its consequent '$5 \cdot 0 = 7 \cdot 0$'.

There is a similar fallacy related to modus tollens. It is called <u>denying</u> <u>the</u> <u>antecedent</u>.

[denying the antecedent]

INVALID

$$\frac{\text{not } p \qquad \text{if } p \text{ then } q}{\text{not } q}$$

[modus tollens]

$$\frac{\text{if } p \text{ then } q \qquad \text{not } q}{\text{not } p}$$

An example of this fallacy is inferring that $5 \cdot 0 \neq 7 \cdot 0$ from the conditional sentence 'if $5 = 7$ then $5 \cdot 0 = 7 \cdot 0$' and the denial, '$5 \neq 7$', of its antecedent. Here is another example in which this kind of invalid reasoning is used:

> Each solution of the inequation '$x < 3$' is a solution of '$x < 5$'. Since 4 does not satisfy '$x < 3$', it follows that 4 does not satisfy '$x < 5$'.

* * *

<u>B.</u> Each of the following exercises gives a pattern for a sequence of inferences. Write sentences which complete the pattern.

1. $$\cfrac{\cfrac{a \overset{*}{=} 2 \qquad a^2 - 3a - 6 = 0}{2^2 - 3 \cdot 2 - 6 = 0} \; * \qquad 2^2 - 3 \cdot 2 - 6 \neq 0}{}$$

2. $$\cfrac{\cfrac{\text{if } a = 0 \text{ then } ab = 0 \qquad ab \overset{*}{\neq} 0}{} \qquad \text{if } a \neq 0 \text{ then } a^2 > 0}{\rule{3cm}{0.4pt}} \; *$$

3. $$\cfrac{\text{if } A \in \overline{BC} \text{ then } A \in \overleftrightarrow{BC} \qquad A \overset{*}{\notin} \overleftrightarrow{BC}}{\rule{3cm}{0.4pt}} \; *$$

4. $$\cfrac{\text{if } p \text{ then } q \qquad \overset{*}{\text{not}} \; q}{\rule{3cm}{0.4pt}} \; *$$

C. Here is a proof of the algebra theorem:

$$\forall_x \forall_y \text{ if } -x \neq y \text{ then } x + y \neq 0$$

The proof illustrates the use of modus tollens.

(1)	$-a \neq b$	[assumption] *
(2)	$\forall_x \forall_y$ if $x + y = 0$ then $-x = y$	[theorem]
(3)	if $a + b = 0$ then $-a = b$	[(2)]
(4)	$a + b \neq 0$	[(1) and (3)]
(5)	if $-a \neq b$ then $a + b \neq 0$	[(4); * (1)]
(6)	$\forall_x \forall_y$ if $-x \neq y$ then $x + y \neq 0$	[(1) - (5)]

1. Make a diagram of this proof.

2. Tell the rule of reasoning which justifies inferring

   (a) (3) from (2) (b) (4) from (1) and (3)

   (c) (5) from (4) (d) (6) from steps (1) through (5)

3. (a) Of what premisses is (4) a consequence?

   (b) Of what premisses is (5) a consequence?

   (c) Of what premisses is (6) a consequence?

   (d) Why is (6) a theorem?

4. Compare steps (3), (1), and (5) with your solution of Exercise 4 of Part B on page 6-379. Do you see that (5) is a consequence of (3)?

CONTRAPOSITION

In Exercise 4, Part C, you noticed that the conditional sentence:

(3) if $a + b = 0$ then $-a = b$

implies the conditional sentence:

(5) if $-a \neq b$ then $a + b \neq 0$

The justification for this can be seen by examining the proof on page 6-380. By modus tollens, (3) together with the assumption '$-a \neq b$' [(1)] imply '$a + b \neq 0$' [(4)]. Conditionalizing, and discharging the assumption gives us (5).

The conditional sentence (5) is said to be the <u>contrapositive</u> of (3). We have seen that the conditional sentence (3) implies its contrapositive.

Is it the case that each conditional sentence implies its contrapositive? In other words, is the inference scheme:

if p then q
if not q then not p

a valid one? If you did Exercise 4 on page 6-379, you saw that each inference of this kind can be justified in the same way we justified the inference of (5) from (3):

$$\dfrac{\dfrac{\text{if p then q} \qquad \overset{*}{\text{not q}}}{\text{not p}}\ [\text{modus tollens}]}{\text{if not q then not p}}\ [\text{conditionalizing}]\ *$$

So, we have justified a new rule of reasoning--<u>the rule of contraposition</u>. By using this rule, we could have moved directly in the proof on page 6-380 from step (3) to step (5).

Let's make use of this new rule in proving another algebra theorem:

$$\forall_x \forall_y \text{ if } xy \neq 0 \text{ then } y \neq 0$$

The next-to-last step in such a proof will be a conditional sentence, say:

$$\text{if } ab \neq 0 \text{ then } b \neq 0$$

The rule of contraposition suggests the following strategy: Try to derive the conditional sentence:

$$\text{if } b = 0 \text{ then } ab = 0,$$

and then use contraposition. So, now our job is to derive 'ab = 0' from the assumption 'b = 0'.

(1)	$b = 0$	[assumption] *
(2)	$\forall_x\, x0 = 0$	[theorem]
(3)	$a0 = 0$	[(2)]
(4)	$ab = 0$	[(1) and (3)]
(5)	if $b = 0$ then $ab = 0$	[(4); * (1)]
(6)	if $ab \neq 0$ then $b \neq 0$	[from (5) by contraposition]
(7)	$\forall_x \forall_y$ if $xy \neq 0$ then $y \neq 0$	[(1) - (6)]

THE RULES OF DOUBLE DENIAL

Here are three sentences:

(1) $1 = 2$ (2) $x < y$ (3) $A \in \ell$

Here is one way of writing denials of these sentences:

$1 \neq 2$ $x \not< y$ $A \notin \ell$,

and here is another way:

(4) not $(1 = 2)$ (5) not $(x < y)$ (6) not $(A \in \ell)$

Now, consider the denials of sentences (4), (5), and (6):

(7) not not $(1 = 2)$ (8) not not $(x < y)$ (9) not not $(A \in \ell)$

There is a rule of reasoning which tells us that (7) is a consequence of (1), that (8) is a consequence of (2), and that (9) is a consequence of (3); and there is another rule that tells us that (1), (2), and (3) are consequences of (7), (8), and (9), respectively. These two rules tell what people mean when they say that two 'no's are a 'yes'.

The first of these rules is called <u>the rule of double denial</u>:

Double denial.
$\dfrac{p}{\text{not not } p}$

and the second is called <u>the reverse rule of double denial:</u>

Reverse double denial.
$\dfrac{\text{not not } p}{p}$

One of the important uses of the rules of double denial is in providing us with more techniques for deriving conditional sentences. For example, suppose we are given the premisses:

(1) $a = b$

and: (2) if $a > b$ then $a \neq b$ [that is: not $(a = b)$]

Now, by the rule of double denial, (1) implies:

(3) not not $(a = b)$ [that is: not $(a \neq b)$]

which is the denial of the consequent of (2). So, by modus tollens, (2) and (3) imply:

(4) $a \not> b$ [that is, not $(a > b)$]

So, (4) is a consequence of (1) and (2). Going a step further, we can conditionalize (4) and discharge (1) to get:

(5) if $a = b$ then $a \not> b$

So, (5) is a consequence of (2).

Here is an outline of the reasoning:

$$\cfrac{(2) \qquad \cfrac{\overset{*}{(1)}}{(3)}}{\cfrac{(4)}{(5)}\,*}$$

The procedure by which we inferred (5) from (2) shows that a conditional sentence [like (2)] of the form:

if p then not q

implies the corresponding conditional sentence [(5)] of the form:

if q then not p

We shall call this a symmetric rule of contraposition:

$$\boxed{\dfrac{\text{if p then not q}}{\text{if q then not p}}}$$

Using this rule, we could have inferred (5) directly from (2).

There are two other rules of contraposition which you will discover in doing the exercises which follow.

EXERCISES

A. Write the contrapositive of each conditional sentence.

1. if $A \in \overline{BC}$ then $A \in \overrightarrow{BC}$

2. if $A \in \overleftrightarrow{BC}$ then $A \in \overline{BC}$

3. if $a = 2$ then $a^2 = 4$

4. $a^2 = 9$ if $a = 3$ [This is a translation of 'if $a = 3$ then $a^2 = 9$'.]

5. $a^2 = 9$ only if $a = 3$ [This is a translation of 'if $a^2 = 9$ then $a = 3$'.]

B. Write sentences to complete the patterns and tell what rules justify the inferences.

1. (a)

$$\dfrac{\text{if } A \in \overline{BC} \text{ then } B \neq C \qquad \dfrac{B \overset{*}{=} C}{\text{not } (B \neq C)}}{\rule{4cm}{0.4pt}}\ *$$

(b)

$$\dfrac{\text{if p then not q} \qquad \dfrac{\overset{*}{\text{q}}}{\text{not not q}}}{\rule{4cm}{0.4pt}}\ *$$

2. (a) if $A \neq B$ then $B \in \overrightarrow{AB}$ $\qquad$ * $B \notin \overrightarrow{AB}$

$\qquad$ ____________

$\qquad$ $A = B$ *

(b) if not p then q $\qquad$ * not q

$\qquad$ ____________

$\qquad$ p

3. (a) $\qquad$ * $A \in \overline{BC}$

if $A \notin \overleftrightarrow{BC}$ then $A \notin \overline{BC}$

$\qquad$ not $(A \notin \overleftrightarrow{BC})$

$\qquad$ ____________________ *

(b) $\qquad$ * p

if not q then not p

$\qquad$ not not q

$\qquad$ ______________ *

4. (a) if $A \in \overline{BC}$ then $A \in \overleftrightarrow{BC}$ $\qquad$ * $A \notin \overleftrightarrow{BC}$

$\qquad$ ________________________ *

(b) if p then q $\qquad$ * not q

$\qquad$ ____________ *

THE FOUR RULES OF CONTRAPOSITION

In Exercise 1(b) of Part B, above, you justified inferences of the form:

$$\frac{\text{if } p \text{ then not } q}{\text{if } q \text{ then not } p}$$

And, on page 6-384, we called this <u>a symmetric rule of contraposition</u>.

In Exercise 2(b), you established the other symmetric rule of contraposition by justifying inferences of this form:

$$\frac{\text{if not } p \text{ then } q}{\text{if not } q \text{ then } p}$$

And, in Exercise 3(b), you established <u>the reverse rule of contraposition</u> by justifying inferences of the form:

$$\frac{\text{if not } p \text{ then not } q}{\text{if } q \text{ then } p}$$

These rules of reasoning together with <u>the rule of contraposition</u>, which you re-established in Exercise 4(b):

$$\frac{\text{if } p \text{ then } q}{\text{if not } q \text{ then not } p}$$

provide us with four ways of transforming conditional sentences. In referring to any of these inferences when explaining the reasoning in a proof, all you need say is 'contraposition'.

Complete each of the following to a true sentence:

$$\frac{\text{if } r = s \text{ then } w = t}{\boxed{}}$$ is an instance of the rule of contraposition.

$$\frac{\text{if } r \neq s \text{ then } w \neq t}{\boxed{}}$$ is an instance of the reverse rule of contraposition.

CONVERSES

There are two fallacies which are somewhat like the four kinds of contraposition:

if p then q
―――――
if q then p
INVALID
[inferring the converse]

and:

if p then q
―――――
if not p then not q
INVALID
[inferring the inverse]

Each is closely related to one of the two fallacies pointed out earlier:

if p then q   q
―――――
p
INVALID
[affirming the consequent]

and:

not p   if p then q
―――――
not q
INVALID
[denying the antecedent]

People who commit the fallacy of affirming the consequent probably do so because they think [unjustifiably] that a conditional sentence implies its converse. [The converse of a conditional is what you get when you interchange the antecedent and the consequent.] For example, they may think that the conditional sentence:

if John lives in Seattle then John lives in Washington

implies its converse:

if John lives in Washington then John lives in Seattle

[Of course, the first doesn't imply the second. Since Seattle is in Washington, the first sentence is certainly true, while the second sentence may be false. It is false if John lives in Tacoma.]

Although the word 'converse' can be applied properly only to sentences which are conditional sentences, it is customary to use it, also, in connection with universal generalizations of conditional sentences. Thus, the uniqueness principle for addition:

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } x + z = y + z$$

may be described as the converse of the cancellation principle for addition:

$$\forall_x \forall_y \forall_z \text{ if } x + z = y + z \text{ then } x = y$$

And, of course, the cancellation principle for addition may be described as the converse of the uniqueness principle for addition. [But, strictly speaking, only corresponding instances of these principles are converses of one another.]

Similar remarks apply to the use of 'contrapositive'. Although this word, like 'converse', applies properly only to conditional sentences, it may also be applied to universal generalizations of such sentences. Then, one may say that:

$$\forall_x \forall_y \forall_z \text{ if } x + z \neq y + z \text{ then } x \neq y$$

is the <u>contrapositive</u> of the uniqueness principle for addition. In this sense of the word, it is still the case that a sentence implies, and is implied by, its contrapositive.

EXERCISES

<u>A.</u> Each exercise contains a conditional sentence or a universal generalization of a conditional sentence. Write the contrapositive and the converse.

1. if $A \in \overrightarrow{BC}$ then $A \in \overset{\bullet\!\!\longrightarrow}{BC}$

   Contrapositive: ______________________________

   Converse: ______________________________

2. if $A \in \overset{\bullet\!\!\longrightarrow}{BC}$ then $A \in \overrightarrow{BC}$

   Contrapositive: ______________________________

   Converse: ______________________________

3. if $A = B$ then $\overline{AB} = \emptyset$

   Contrapositive: ______________________________

   Converse: ______________________________

4. $ac = bc$ if $a = c$

   Contrapositive: ______________________________

   Converse: ______________________________

5. $ac = bc$ only if $a = c$

   Contrapositive: ______________________________

   Converse: ______________________________

6. $\forall_X \forall_Y \forall_Z$ if $Z \in \overleftrightarrow{XY}$ then X, Y, and Z are collinear

Contrapositive: ____________________

Converse: ____________________

7. $\forall_x \forall_y$ if $x + y = 0$ then $-x = y$

Contrapositive: ____________________

Converse: ____________________

8. $\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}$ if $\frac{x}{y} = \frac{u}{v}$ then $xv = uy$

Contrapositive: ____________________

Converse: ____________________

9. $\forall_X \forall_Y$ if $\{Z: Y \in \overline{XZ}\} = \emptyset$ then $X = Y$

Contrapositive: ____________________

Converse: ____________________

B. Which of the following inferences are examples of fallacies?

1. $\dfrac{a \neq b \qquad \text{if } a = b \text{ then } ac = bc}{ac \neq bc}$

2. $\dfrac{a = b \qquad \text{if } a = b \text{ then } ac = bc}{ac = bc}$

3. $\dfrac{a \neq b \qquad \text{if } ac = bc \text{ then } a = b}{ac \neq bc}$

4. $\dfrac{a = b \qquad \text{if } a - b \neq 0 \text{ then } a \neq b}{a - b = 0}$

5. $\dfrac{\text{if } \ell \text{ is parallel to } m \text{ then } \ell \neq m}{\text{if } \ell = m \text{ then } \ell \text{ is not parallel to } m}$

6. $\dfrac{\text{if } \ell \text{ is parallel to } m \text{ then } \ell \neq m}{\text{if } \ell \text{ is not parallel to } m \text{ then not } (\ell \neq m)}$

7. $\dfrac{\text{if } a = b \text{ then } a^2 = b^2}{\text{if } a^2 = b^2 \text{ then } a = b}$

8. $\dfrac{\text{if } a = b \text{ then } a - b = 0}{\text{if } a - b = 0 \text{ then } a = b}$

9. $\dfrac{\text{if } A \neq B \text{ then } \overline{AB} \neq \emptyset \qquad \overline{AB} \neq \emptyset}{A \neq B}$

10. $\dfrac{\overline{AB} \neq \emptyset \qquad \text{if } \overline{AB} \neq \emptyset \text{ then } A \neq B}{A \neq B}$

BICONDITIONAL SENTENCES

In Unit 2 you proved the uniqueness principle for addition:

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } x + z = y + z,$$

and then you proved its converse, the cancellation principle for addition:

$$\forall_x \forall_y \forall_z\ x + z = y + z \text{ then } x = y$$

In Unit 3, you combined these two theorems into the addition transformation principle:

$$\forall_x \forall_y \forall_z\ x + z = y + z\ \underbrace{\text{if}}_{\text{uniqueness prin.}} \text{ and } \underbrace{\text{only if}}_{\text{cancellation prin.}}\ x = y$$

A sentence of the form:

p if and only if q

is called a <u>biconditional</u> <u>sentence</u>. The reason for using the word 'biconditional' is that such a sentence combines <u>two</u> conditional sentences. Each of the conditional sentences being combined is the converse of the other. It is a rule of logic that a biconditional sentence is a consequence of the two conditionals of which it is composed, and also that it implies each of them. Schematically:

<u>Biconditionals.</u>

$$\frac{\text{if q then p} \qquad \text{if p then q}}{\text{p if and only if q}}$$

$$\frac{\text{p if and only if q}}{\text{if q then p}} \qquad \frac{\text{p if and only if q}}{\text{if p then q}}$$

For example, an instance of Axiom 6 on page 6-20 is a biconditional sentence:

$C \in \overleftrightarrow{AB}$ if and only if $A \neq B$ and A, B, and C are collinear

From this sentence we can infer two conditionals:

if $A \neq B$ and A, B, and C are collinear then $C \in \overleftrightarrow{AB}$

and:

if $C \in \overleftrightarrow{AB}$ then $A \neq B$ and A, B, and C are collinear

Biconditional sentences are important because they can be used in simplifying other sentences. Here is an example of how you did this in algebra. Suppose you want to solve the quadratic equation:

$$(1) \quad 21x^2 - 29x - 10 = 0$$

In the process of doing so, you factor the left side of (1) and then transform (1) into the equivalent sentence:

$$(2) \quad 3x - 5 = 0 \text{ or } 7x + 2 = 0,$$

Then you transform (2) into the equivalent sentence:

$$(3) \quad x = \frac{5}{3} \text{ or } x = -\frac{2}{7}$$

What are the logical grounds for claiming that (2) and (3) are equivalent? Using the two equation transformation principles, you can derive the following two biconditionals:

$$(a) \quad 3x - 5 = 0 \text{ if and only if } x = \frac{5}{3}$$

$$(b) \quad 7x + 2 = 0 \text{ if and only if } x = -\frac{2}{7}$$

We can derive (3) from (2) [and (2) from (3) for that matter] by using what is called <u>the substitution rule for biconditionals</u>. Consider (a) and (2):

$$(a) \quad 3x - 5 = 0 \text{ if and only if } x = \frac{5}{3}$$

$$(2) \quad 3x - 5 = 0 \quad \text{or} \quad 7x + 2 = 0$$

We substitute the right side ['$x = \frac{5}{3}$'] of (a) for its left side ['$3x - 5 = 0$'] in (2) to obtain:

$$(2.1) \quad x = \frac{5}{3} \quad \text{or} \quad 7x + 2 = 0$$

The substitution rule for biconditionals tells us that (2.1) follows from (a) and (2).

Now, consider (b) and (2.1). Use the substitution rule to show that these sentences imply (3).

Read the substitution rule for equations in the box on page 6-359. Using this as a pattern, formulate the substitution rule for biconditional sentences.

CONJUNCTION SENTENCES

A biconditional sentence--one of the form:

r if and only if s

can be thought of as an abbreviation for a sentence of the form:

[r if s] *and* [r only if s]

or:

[if s then r] *and* [if r then s]

A sentence of the form:

p and q

is called *a conjunction sentence*. So, for example, a biconditional sentence is an abbreviation for a special kind of conjunction sentence. Moreover, the three logical principles for biconditional sentences given on page 6-390 are consequences of three logical principles for conjunction sentences:

Conjunctions.

$$\frac{p \qquad q}{p \text{ and } q}$$

$$\frac{p \text{ and } q}{p} \qquad\qquad \frac{p \text{ and } q}{q}$$

If, in each of the schemes above, you replace 'p' and 'q' by a conditional sentence and its converse, and then abbreviate the conjunction sentence you obtain to the corresponding biconditional sentence, you will have an inference of one of the three basic kinds for biconditional sentences.

Evidently, if you want to derive a conjunction sentence from given premisses, all you need do is derive each of its two parts from the given premisses, and then use an inference of the first of the three kinds. On the other hand, if you make use of two premisses in deriving a conclusion you can use inferences of the second and third kinds to show that the conclusion is a consequence of the conjunction of the two premisses.

As an illustration of how inferences of these kinds work, let's use

them in justifying two useful rules of reasoning:

Importation.	Exportation.
$\dfrac{\text{if p then [if q then r]}}{\text{if [p and q] then r}}$	$\dfrac{\text{if [p and q] then r}}{\text{if p then [if q then r]}}$

$$\cfrac{\cfrac{\cfrac{\overset{*}{\text{p and q}}}{\text{q}} \qquad \cfrac{\cfrac{\overset{*}{\text{p and q}}}{\text{p}} \quad \text{if p then [if q then r]}}{\text{if q then r}}}{\text{r}}\ *}{\text{if [p and q] then r}}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\overset{\dagger}{\text{p}} \quad \overset{*}{\text{q}}}{\text{p and q}} \quad \text{if [p and q] then r}}{\text{r}}\ *}{\text{if q then r}}\ \dagger}{\text{if p then [if q then r]}}$$

ALTERNATION SENTENCES

A sentence of the form:

p or q

is called <u>an alternation sentence</u>.

As was the case for conjunction sentences, there are three basic rules of reasoning for alternation sentences. Schematically:

Alternations.

$$\frac{\text{p}}{\text{p or q}} \qquad\qquad \frac{\text{q}}{\text{p or q}}$$

$$\frac{\text{p or q} \qquad \text{if p then r} \qquad \text{if q then r}}{\text{r}}$$

In algebra when you solve a quadratic equation and get to the equivalent sentence, say, '$x = 3$ or $x = 4$', you conclude that 4 is one of the solutions. Your conclusion is justified by the second of these three basic rules for alternation sentences. That is, the sentence '$x = 4$' implies '$x = 3$ or $x = 4$'.

The third basic rule is called <u>the rule of the dilemma.</u> If you have only two choices, and whichever choice you make the same thing will happen, then you can't avoid its happening. [You are faced with a dilemma.] [This rule is used in arriving at the last step in the proof on page 6-91.]

The rule of the dilemma is used in proving the algebra theorem:

$$\forall_x \forall_y \text{ if } x = 0 \text{ or } y = 0 \text{ then } xy = 0$$

(1)	$a = 0$ or $b = 0$	[assumption] *
(2)	$\forall_x \forall_y$ if $x = 0$ then $xy = 0$	[theorem]
(3)	if $a = 0$ then $ab = 0$	[(2)]
(4)	if $b = 0$ then $ba = 0$	[(2)]
(5)	$\forall_x \forall_y$ $xy = yx$	[basic principle]
(6)	$ab = ba$	[(5)]
(7)	if $b = 0$ then $ab = 0$	[(4) and (6)]
(8)	$ab = 0$	[(1), (3), and (7)]
(9)	if $a = 0$ or $b = 0$ then $ab = 0$	[(8); * (1)]
(10)	$\forall_x \forall_y$ if $x = 0$ or $y = 0$ then $xy = 0$	[(1) - (9)]

What step is justified by the rule of the dilemma? Try to diagram this proof.

There is another principle of logic, sometimes called <u>the law of the excluded middle</u>, which says that any consequence of premisses one of which is of the form:

p or not p

is a consequence of the other premisses alone. So, in a sense, premisses of this form can be thought of as discharged just as soon as you write them in a proof. [This rule is used in the proof on page 6-47.]

Finally, there is a rule of reasoning which may be called <u>the rule for denying an alternative</u>, which can be expressed schematically by:

$$\frac{p \text{ or } q \quad \text{not } q}{p} \qquad\qquad \frac{p \text{ or } q \quad \text{not } p}{q}$$

[This rule is used in the proof on pages 6-44 and 6-45.]

SUMMARY OF KINDS OF VALID INFERENCE

We have discussed some of the correct ways of inferring conclusions from premisses. When you are writing proofs, the rules which describe these ways of drawing conclusions will help you decide what steps to put in to arrive at the desired conclusion. On the other hand, when you are studying a proof, or when you are explaining one to other people, you may need to refer to these rules in order to see, or to show, why a certain step follows from preceding steps.

Here are inference schemes which we have considered.

Substitution rule for equations.

$$\frac{[\text{equation}] \qquad \text{sentence (1)}}{\text{sentence (2)}}$$, where sentence (2) is obtained by replacing one side of the equation, somewhere in sentence (1), by the other side of the equation.

Universal instantiation.

$$\frac{[\text{a universal generalization}]}{[\text{any one of its instances}]}, \quad \text{for example:} \quad \frac{\forall_x \forall_y\ x + y = y + x}{3 + 2 = 2 + 3}$$

Modus ponens.

$$\frac{p \qquad \text{if } p \text{ then } q}{q}$$

Modus tollens.

$$\frac{\text{if } p \text{ then } q \qquad \text{not } q}{\text{not } p}$$

Conditionalizing, and discharging an assumption.

$$\frac{\begin{matrix}[p] \\ q\end{matrix}}{\text{if } p \text{ then } q}$$

Hypothetical syllogism. [Ex. 6 on page 6-377].

$$\frac{\text{if } p \text{ then } q \qquad \text{if } q \text{ then } r}{\text{if } p \text{ then } r}$$

Contraposition.

$$\frac{\text{if p then q}}{\text{if not q then not p}} \qquad \frac{\text{if not q then not p}}{\text{if p then q}}$$

$$\frac{\text{if p then not q}}{\text{if q then not p}} \qquad \frac{\text{if not p then q}}{\text{if not q then p}}$$

Double denial.

$$\frac{\text{p}}{\text{not not q}} \qquad \frac{\text{not not p}}{\text{p}}$$

Biconditionals.

$$\frac{\text{if q then p} \qquad \text{if p then q}}{\text{p if and only if q}}$$

$$\frac{\text{p if and only if q}}{\text{if q then p}} \qquad \frac{\text{p if and only if q}}{\text{if p then q}}$$

Substitution rule for biconditionals.

$$\frac{\text{[biconditional]} \qquad \text{sentence (1)}}{\text{sentence (2)}},$$

where sentence (2) is obtained by replacing one side of the biconditional, somewhere in sentence (1), by the other side of the biconditional.

Conjunctions.

$$\frac{\text{p} \qquad \text{q}}{\text{p and q}} \qquad \frac{\text{p and q}}{\text{p}} \qquad \frac{\text{p and q}}{\text{q}}$$

Importation and Exportation.

$$\frac{\text{if p then [if q then r]}}{\text{if [p and q] then r}} \qquad \frac{\text{if [p and q] then r}}{\text{if p then [if q then r]}}$$

Alternations.

$$\frac{p}{p \text{ or } q} \qquad \frac{q}{p \text{ or } q} \qquad \frac{p \text{ or } q \quad \text{if } p \text{ then } r \quad \text{if } q \text{ then } r}{r}$$

Denying an alternative.

$$\frac{p \text{ or } q \quad \text{not } q}{p} \qquad \frac{p \text{ or } q \quad \text{not } p}{q}$$

In addition to these kinds of inferences, we have used:

The test-pattern principle.

An universal generalization sentence is justified by a test-pattern for its instances.

The principle of identity.

A thing is equal to itself.

The law of the excluded middle.

p or not p

MISCELLANEOUS NOTES

The foregoing discussion on logic provides you with a good beginning for the task of understanding and writing proofs. As you go through the sections of the geometry course, you will encounter many uses of these rules of reasoning. Most of these have been discussed in the preceding pages of the Appendix.

In these notes we shall discuss certain logical aspects of various geometry proofs.

*

The first of these is the proof of Theorem 1-1 given on page 6-33 and discussed further on pages 6-34 and 6-35.

Suppose we diagram this proof to show its logical structure.

$$\begin{array}{c} \begin{matrix} \cdots \\ (3) \end{matrix} \quad \dfrac{(1)}{(2)} \\ \hline (4) \\ \cdots \\ (5) \\ \hline\hline (6) \end{array}$$

Notice the dotted bars. These indicate missing steps, or gaps in the proof. If the missing steps had been included in the proof, the diagram would look like this:

$$\begin{array}{c} \dfrac{\dfrac{(2.1)}{(3)} \quad \dfrac{(1)}{(2)}}{(4)} \qquad \dfrac{(4.1)}{(4.2)} \\ \hline (5) \\ \hline\hline (6) \end{array}$$

The outline of the shorter proof shows that (6) [that is, Theorem 1-1] is a consequence of (1) [Axiom A] together with some Introduction theorems or algebra theorems. The other outline shows that these theorems are (2.1) and (4.1).

The proof of Theorem 1-2 on page 6-36.

1. Give the rule of reasoning which justifies each of the inferences listed below.

   (a) (3) from (2) ____________________

   (b) (5) from (3) and (4) ____________________

   (c) (7) from (6) ____________________

   (d) (8) from (5) and (7) ____________________

   (e) (11) from (10) and discharging (1) ____________________

   (f) (12) from (1) - (11) ____________________

2. Step (9) follows from (8) and an instance of an Introduction Theorem. What instance? What rule of reasoning justifies this inference?

3. Here is the start of an outline of the proof of Theorem 1-2. Finish it.

   (1)
   . . .
   (4)
   ____

4. The marginal comment for (4) shows that it is a consequence of (1) and an Introduction theorem. There are two such theorems either of which might be used. One is:

   (*) $\forall_X \forall_Y$ if $Y \in \overleftrightarrow{XX}$ then $X = Y$

   and the other is its contrapositive.

   (a) From (1) and an instance of (*), you could infer (4). What rule of reasoning justifies this?

   (b) From (1) and an instance of the contrapositive of (*), you could infer (4). What rule of reasoning justifies this?

The proof of Theorem 1-5 on pages 6-44 and 6-45.

This proof illustrates the use of the rule for denying an alternative [page 6-394], as well as the basic rule for deriving a biconditional sentence [page 6-390].

We are trying to derive a conditional like:

if $C \in \overrightarrow{AB}$ then [$C \in \overline{AB}$ if and only if $AC < AB$]

So, the natural prodedure is to take '$C \in \overrightarrow{AB}$' as an assumption. From this assumption together with theorems [including perhaps axioms and logical principles], we derive the consequent which is a biconditional:

$C \in \overline{AB}$ if and only if $AC < AB$

This will be easy to do if we can derive the two conditionals:

if $AC < AB$ then $C \in \overline{AB}$, and: if $C \in \overline{AB}$ then $AC < AB$

The second of these is an instance of Theorem 1-3. So, all we have to do is derive the first.

Let's formalize the discussion on pages 6-44 and 6-45.

We start with the assumption:

(1) $C \in \overrightarrow{AB}$

Now, by an Introduction Theorem, it follows that

(2) if $C \in \overrightarrow{AB}$ then [$C \in \overline{AB}$ or $C = B$ or $B \in \overline{AC}$].

By modus ponens, (1) and (2) give us:

(3) $C \in \overline{AB}$ or $C = B$ or $B \in \overline{AC}$

Next, we introduce the assumption:

(4) $AC < AB$

By algebra, (4) implies:

(5) $AC \not> AB$

Also, by Theorem 1-3, we have:

(6) if $B \in \overline{AC}$ then $AC > AB$

By modus tollens, (5) and (6) give us:

(7) $B \notin \overline{AC}$

By the rule for denying an alternative, (3) and (7) give us:

(8) $C \in \overline{AB}$ or $C = B$

By algebra, again, (4) implies:

(9) $AC \neq AB$

By the principle of identity, we have:

(10) if $C = B$ then $AC = AB$

By modus tollens, (9) and (10) give us:

(11) $C \neq B$

By the rule for denying an alternative, (8) and (11) give us:

(12) $C \in \overline{AB}$

Conditionalizing (12), and discharging the assumption (4), we obtain:

(13) if $AC < AB$ then $C \in \overline{AB}$

Combining this with the following instance of Theorem 1-3:

(14) if $C \in \overline{AB}$ then $AC < AB$

gives us, by the basic rule for deriving a biconditional sentence:

(15) $C \in \overline{AB}$ if and only if $AC < AB$

Finally, conditionalizing (15), and discharging the assumption (1) results in:

(16) if $C \in \overrightarrow{AB}$ then [$C \in \overline{AB}$ if and only if $AC < AB$]

SUPPLEMENTARY EXERCISES

Page 6-9.

1. True or false?

(a) $5 \in \{3, 8, 6, 5, 7\}$

(b) $10 \in \{x: 2x + 1 < 20\}$

(c) $2 \in \{y: \text{ y is a prime}\}$

(d) Dallas $\in$ {x, x is a city: x is in Texas}

(e) $\{1, 5, 6, 8\} \cap \{2, 4, 5, 6, 9\} = \{5, 6\}$

(f) $\{1, 5, 6, 8\} \cup \{2, 4, 5, 6, 9\} = \{1, 2, 4, 5, 6, 8, 9\}$

(g) $\{3, 8, 7\} \cap \{4, 8, 7\} = \{8\}$

(h) $\{1, 2, 3\} \cap \{4, 5, 6\} = \emptyset$

(i) There is a number x such that $x \in \{1, 2, 3\} \cap \{4, 5, 6\}$.

(j) There is a number x such that $x \in \{1, 2, 3\} \cup \{4, 5, 6\}$.

(k) $A \in \{A, B, C\} \cap \{A, B, D\}$

(ℓ) $\{A, B, C\} \cup \{A, B, D\} = \{A, B, C, D\}$

(m) $\{2, 3\} \subseteq \{1, 2, 3, 4, 5\}$

(n) $\{5, 8, 10\} \subseteq \{5, 8, 12, 15\}$

(o) $\{8, 11\} \subseteq \{8, 11\}$

(p) $\emptyset \subseteq \{1, 2, 3\}$

(q) $\{3, 5, 7, 9\} \cap \{4, 5, 6, 7\} \subseteq \{x: \text{ x is an odd number}\}$

2. Suppose $a = \{1, 2, 3, 4\}$, $b = \{3, 4, 5\}$, and $c = \{5, 6, 7, 8, 9\}$.

(a) $a \cup b = \{$ ______________ $\}$ (b) $a \cap b =$ ______________

(c) $a \cup c =$ ______________ (d) $a \cap c =$ ______________

(e) $(a \cup b) \cup c =$ ______________ (f) $(a \cap b) \cup c =$ ______________

(g) $(a \cup b) \cap c =$ ______________ (h) $(a \cap b) \cap c =$ ______________

(i) $a \cup (b \cup c) =$ ______________ (j) $a \cap (b \cap c) =$ ______________

Page 6-16.

Samples.

(1) The complement of $\{2, 3\}$ with respect to $\{1, 2, 3, 4, 5\}$ is $\{1, 4, 5\}$.

(2) The complement of $\{7, 11\}$ with respect to $\{7, 10, 11\}$ is $\{10\}$.

(3) The complement of $\{10\}$ with respect to $\{7, 10, 11\}$ is $\{7, 11\}$.

(4) The complement of $\emptyset$ with respect to $\{1, 2, 3\}$ is $\{1, 2, 3\}$.

(5) The complement of $\{1, 2, 3\}$ with respect to $\{1, 2, 3\}$ is $\emptyset$.

(6) For all sets x and y such that $x \subseteq y$, the complement of x with respect to y is the set of all elements of y which are not in x.

1. Suppose $a = \{1, 2, 3, 4, 5\}$, $b = \{6, 7, 8, 9\}$, and $c = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

(a) The complement of a with respect to c = ______________ .

(b) The complement of b with respect to c = ______________ .

(c) The complement of $(a \cup b)$ with respect to c = __________ .

(d) The complement of $(a \cap b)$ with respect to c = __________ .

(e) The complement of $(a \cap c)$ with respect to c = __________ .

(f) The complement of $(b \cap c)$ with respect to c = __________ .

2. Suppose that h and k are sets and that $h \subseteq k$.

(a) Is the complement of h with respect to k a subset of k?

(b) What is the complement of the complement of h with respect to k?

(c) What is the complement of $(h \cap k)$ with respect to k?

(d) What is the complement of $(h \cup k)$ with respect to k?

3. Suppose that A, B, C, D, and E are five points of a line ℓ, and suppose that they are located as shown in the figure.

(a) $\{X:\ C \in \overline{BX}\} =$ ______ [Answer: $\overrightarrow{CD}$, or: $\overrightarrow{CE}$]

(b) $\{X:\ D \in \overline{EX}\} =$ ______

(c) $\{Y:\ Y \in \overline{CD}\} =$ ______

(d) $\{Z:\ Z = B \text{ or } Z = E \text{ or } Z \in \overline{BE}\} =$ __________

(e) $\{Z:\ Z \in \overline{AB} \text{ or } B \in \overline{AZ}\} =$ __________

(f) $\{X:\ C \in \overline{DX}\} \cup \{X:\ B \in \overline{AX}\} =$ __________

(g) $\{X:\ D \in \overline{CX}\} \cap \{X:\ D \in \overline{EX}\} =$ __________

(h) $\{X:\ X \in \overset{\bullet\!\!\rightarrow}{CD}\} \cap \{X:\ X \in \overset{\bullet\!\!\rightarrow}{CB}\} =$ __________

(i) The complement of $\overrightarrow{CE}$ with respect to $\ell =$ __________.

(j) The complement of $\overset{\bullet\!\!\rightarrow}{CE}$ with respect to $\ell =$ __________.

(k) The complement of $\overline{BC}$ with respect to $\ell = \overset{\bullet\!\!\rightarrow}{BA} \cup$ __________.

(ℓ) The complement of $\overset{\bullet\!-\!\bullet}{CD}$ with respect to $\ell =$ __________.

(m) The complement of $(\overrightarrow{CA} \cup \overset{\bullet\!\!\rightarrow}{CE})$ with respect to $\ell =$ __________.

(n) The complement of $(\overrightarrow{CA} \cup \overrightarrow{CE})$ with respect to $\ell =$ __________.

(o) The complement of $\{B\}$ with respect to $\ell =$ __________.

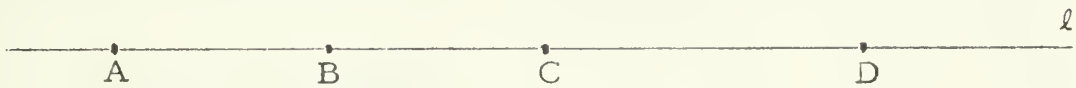
(p) The complement of $\{D\}$ with respect to $\ell =$ __________.

(q) The complement of $\{B, D\}$ with respect to $\ell =$ __________.

(r) The complement of $\{A, B, C, D, E\}$ with respect to $\ell =$ ______.

Page 6-32.

1. Suppose that A, B, C, and D are four points of a line ℓ and arranged as shown in the figure.



(a) If AB = BC and AC = 12 then BA = _____.

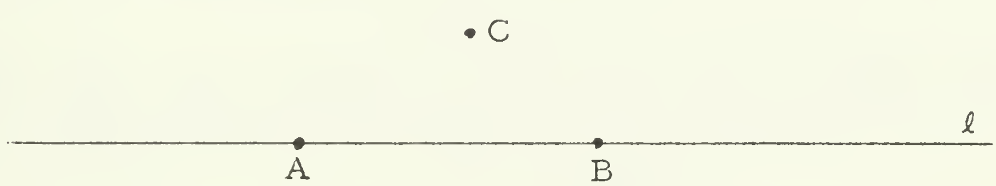
(b) If DC = $2 \cdot$ CB and BD = 15 then BC = _____.

(c) If AB = $\frac{1}{2} \cdot$ BC and BC = $\frac{1}{3} \cdot$ CD then CD = ___ $\cdot$ AB.

(d) If E is a point such that $E \in \ell$ but $E \notin \overleftrightarrow{AC}$, does it follow that CE + ED = CD?

(e) If F is a point such that $F \notin \ell$, show that AF + FC > AB + BC.

2. Suppose that A, B, and C are three points such that A and B are on a line ℓ and C is not.



(a) Tell why there is no point P such that $P \in \overleftrightarrow{AB}$ and AP + PB = AC + CB.

(b) Use your ruler or your compass to locate a point P on ℓ such that AP + PB = AC + CB. [How many such points do you think there are?]

(c) Suppose P_1 and P_2 are two points that fit the conditions described in (b). What can you say about P_1A and P_2B?

☆ 3. (a) Draw a line ℓ and mark on it two points A and B about 2 units [inches] apart. Use your compass and ruler to locate a point P such that AP + PB = 3.

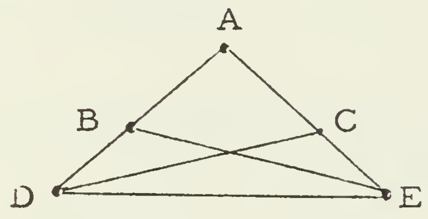
(b) Find other points P which fit the description in (a).

(c) Make a sketch of $\{X\colon\ AX + XB = 3\}$.

Page 6-50.

1. For each exercise, you are given certain data and a figure. State a conclusion which follows and be prepared to justify your conclusion.

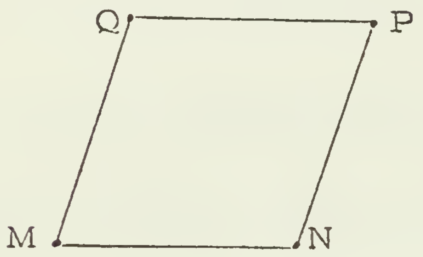
(a)



Given: $AD = AE$,
$BD = CE$

Conclusion: ____________________

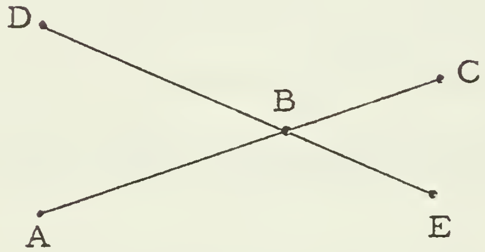
(b)



Given: $MN = QM$,
$PQ = QM$

Conclusion: ____________________

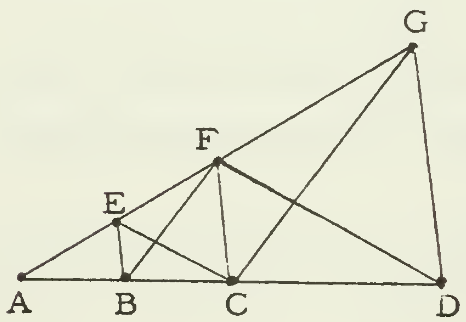
(c)



Given: $DE = AC$,
$BC = BE$

Conclusion: ____________________

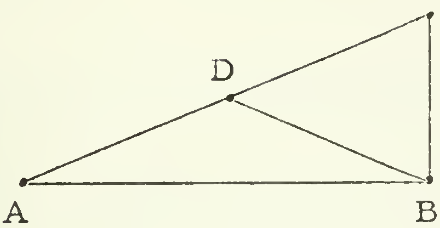
(d)



Given: B is the midpoint of $\overline{AC}$,
E is the midpoint of $\overline{AF}$,
C is the midpoint of $\overline{AD}$,
F is the midpoint of $\overline{AG}$,
$AB = AE$

Conclusion: ____________________

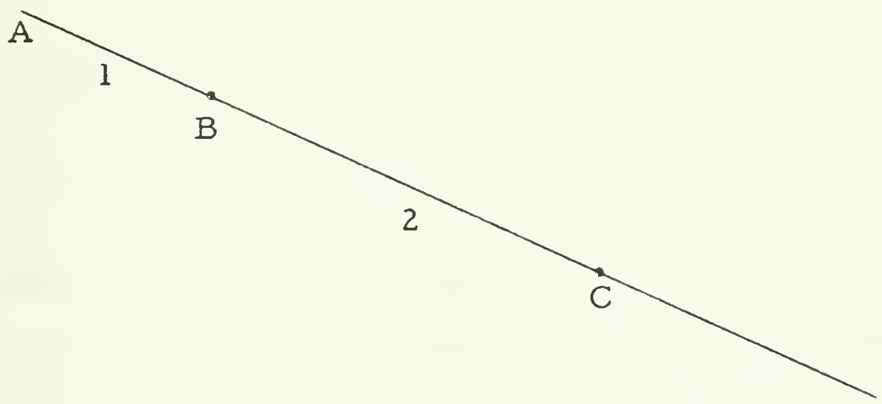
(e)



Given: AD = DB,
BD = DC

Conclusion: ______________

2. Here is a picture of a half-line $\overrightarrow{AB}$ with a point C on it.



If $P \in \overrightarrow{AB}$, let $x(P) = AP$. [Read '$x(P)$' as 'x of P'.] Then, as shown in the figure, $x(B) = 1$, and $x(C) = 3$.

(a) Mark a point D on $\overrightarrow{AB}$ such that $x(D) = 4$. [Use your ruler or compass to do it.]

(b) Mark a point E on $\overrightarrow{AB}$ such that $x(E) > x(C)$ and $x(E) - x(C) = 2$.

(c) Mark a point F on $\overrightarrow{AB}$ such that $x(F) - x(B) = x(C) - x(F)$.

(d) Is it the case that, for each real number k, there is a point $P \in \overrightarrow{AC}$ such that $x(P) = k$? What justifies your answer? Can there be more than one such point P?

(e) How many points P are there such that P belongs to the <u>line</u> $\overleftrightarrow{AB}$ and $AP = 3$?

Page 6-61.

1. (a) Show that an angle is congruent to itself.

   (b) Show that if $\angle A \cong \angle B$ then $\angle B \cong \angle A$.

   (c) Show that if $\angle A \cong \angle B$ and $\angle B \cong \angle C$ then $\angle A \cong \angle C$.

2. Draw $\overrightarrow{AB}$ and locate in one side of $\overleftrightarrow{AB}$ the half-line $\overrightarrow{AC}$ such that $\overrightarrow{AC} \cup \overrightarrow{AB}$ is a right angle.

3. Draw a right angle $\angle MPQ$. Mark a point R in the exterior of $\angle MPQ$ such that $P \in \overline{MR}$. Tell why $\angle RPQ$ is congruent to $\angle MPQ$.

4. Draw a right angle $\angle KJL$. Mark a point Q in the exterior of $\angle KJL$ such that $Q \notin \overleftrightarrow{JL}$ and $Q \notin \overleftrightarrow{JK}$. Mark another point R in the exterior of $\angle KJL$ such that $\angle QJR \cong \angle KJL$. What is $m(\angle RJL) + m(\angle QJK)$?

5. Draw a right angle $\angle ACB$. Locate a point D in the interior of $\angle ACB$ and a point E in the exterior of $\angle ACB$ such that $\angle DCB \cong \angle ECB$ and $\angle DCE$ is a right angle. What are $m(\angle ACD)$ and $m(\angle ACE)$?

6. Draw a right angle $\angle POE$. Locate points M and N in the interior of $\angle POE$ such that $\angle POM \cong \angle MON \cong \angle NOE$. If R is a point such that $O \in \overline{RM}$, what is $m(\angle NOR)$?

7. (a) Draw an angle $\angle ABC$ such that there exists a point D in its interior for which $\angle DBC$ is a right angle. What can you say about the measure of $\angle ABC$?

   (b) Draw an angle $\angle MNP$ such that, for each point X in its interior, $\angle XNP$ is not a right angle. What can you say about the measure of $\angle MNP$?

8. (a) Draw a line $\overleftrightarrow{AB}$, mark a point C on it such that $C \in \overline{AB}$, and mark a point D such that $D \notin \overleftrightarrow{AB}$.

   (b) Locate a point E in the interior of $\angle DCB$ such that $\angle DCE \cong \angle ECB$.

   (c) Locate a point F in the interior of $\angle DCA$ such that $\angle DCF \cong \angle FCA$.

   (d) What is the measure of $\angle ECF$?

Page 6-63.

1. Complete the following table.

$\angle A$	$37°$		$81°$		$(2y)°$	$(90 - x)°$	$[180 - (x+y)]°$
supplement of $\angle A$		$156°$					
complement of $\angle A$				$19°$			

2. Another unit of angle-measure is called the minute. An angle of $1°$ is an angle of $60'$ [60 minutes]. Still another unit of angle-measure is the second. An angle of $1'$ is an angle of $60''$ [60 seconds]. So, for example, an angle of $5\frac{1}{8}°$ is an angle of $5°7\frac{1}{2}'$ or $5°\,7'\,30''$.

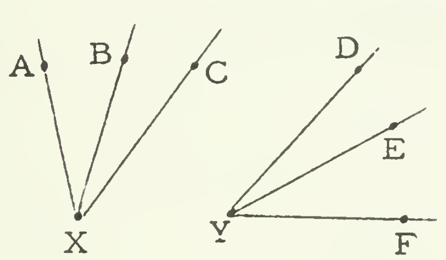
   (a) If D is a point in the interior of $\angle ABC$ such that $\angle ABD$ is an angle of $15°42'35''$ and $\angle DBC$ is an angle of $27°37'48''$, what is the size of $\angle ABC$?

   (b) What is the size of a complement of an angle of $52°18'45''$?

   (c) What is the size of a supplement of an angle of $116°43'18''$?

3. For each exercise, you are given certain data and a figure. State a conclusion which follows and be prepared to justify your conclusion.

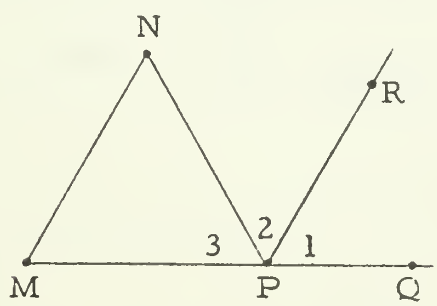
   (a)



Given: $\angle AXB \cong \angle EYF$,
$\angle BXC \cong \angle EYD$

Conclusion: ______________

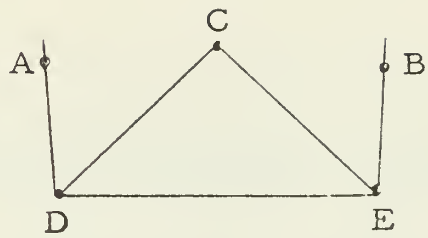
   (b)



Given: $\angle P_1 \cong \angle NMP$,
$\angle P_2 \cong \angle PNM$,
$m(\angle P_1) + m(\angle P_2) + m(\angle P_3) = 180$

Conclusion: ______________

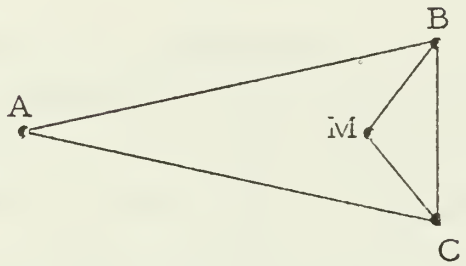
(c)



Given: $\angle ADE \cong \angle BED$,
$\angle ADC \cong \angle BEC$

Conclusion: ____________________

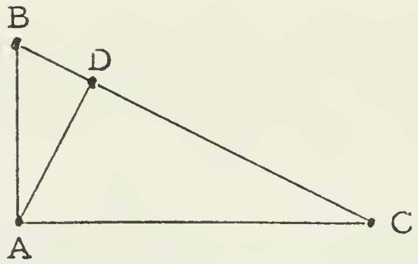
(d)



Given: $\angle ABC \cong \angle ACB$,
$\angle ABM \cong \angle MBC$,
$\angle ACM \cong \angle MCB$

Conclusion: ____________________

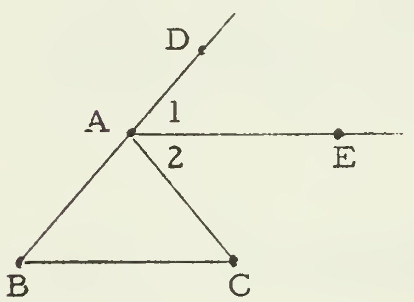
(e)



Given: $\angle BAC$ is a right angle,
$\angle B \cong \angle CAD$

Conclusion: $\angle B$ and $\angle BAD$ are _____

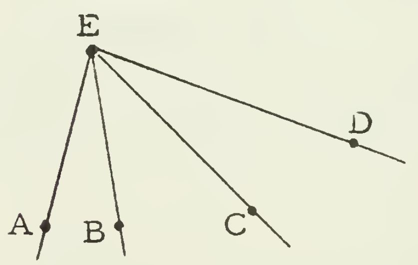
(f)



Given: $\angle A_1 \cong \angle B$,
$\angle A_2 \cong \angle C$,
$\angle A_1 \cong \angle A_2$

Conclusion: ____________________

(g)

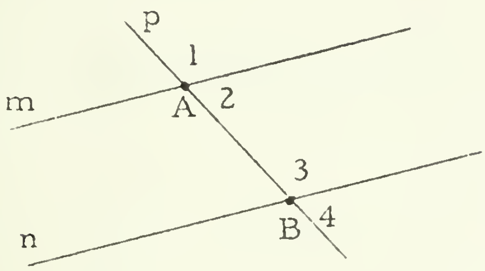


Given: $\angle AEC \cong \angle DEB$

Conclusion: ____________________

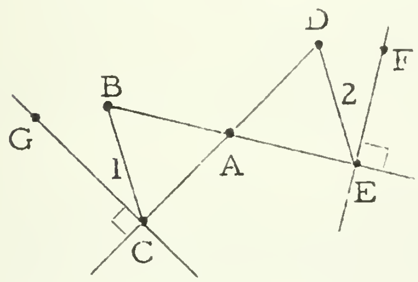
Page 6-74.

For each exercise, you are given certain data and a figure. State a conclusion which follows and be prepared to justify this conclusion.

1. 

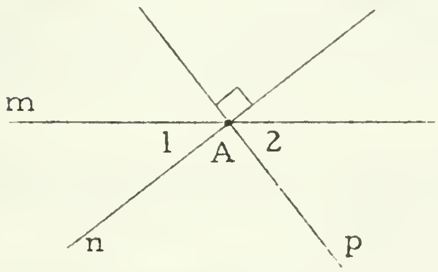
Given: m, n, and p are straight lines, $\angle A_1 \cong \angle B_3$

Conclusion: ____________________

2. 

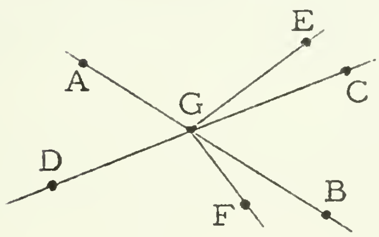
Given: $\overleftrightarrow{EF} \perp \overleftrightarrow{BE}$, $\overleftrightarrow{GC} \perp \overleftrightarrow{CD}$, $\angle D \cong \angle BCA$, $\angle B \cong \angle DEA$, $\angle C_1 \cong \angle E_2$

Conclusion: ____________________

3. 

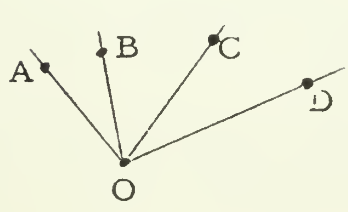
Given: m, n, and p are straight lines, $n \perp p$

Conclusion: ____________________

4. 

Given: $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{G\}$, $\angle CGE \cong \angle BGF$

Conclusion: ____________________

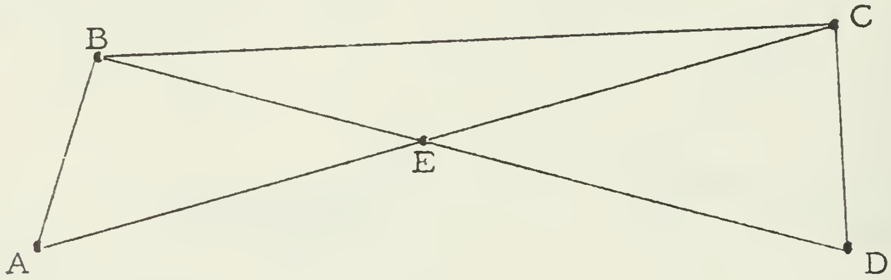
5. 

Given: $\angle AOB \cong \angle DOC$

Conclusion: ____________________

Page 6-81.

1. Listed below are matchings of the vertices of various pairs of triangles pictured in the figure. Tell for which matchings AB and CD are corresponding sides.



(a) ABE $\longleftrightarrow$ DCE	(b) ABE $\longleftrightarrow$ CDE	(c) ABE $\longleftrightarrow$ CED
(d) ABC $\longleftrightarrow$ BCD	(e) ABC $\longleftrightarrow$ CDE	(f) AEB $\longleftrightarrow$ DBC
(g) BCA $\longleftrightarrow$ BCD	(h) BCA $\longleftrightarrow$ CBD	(i) ABE $\longleftrightarrow$ BCD

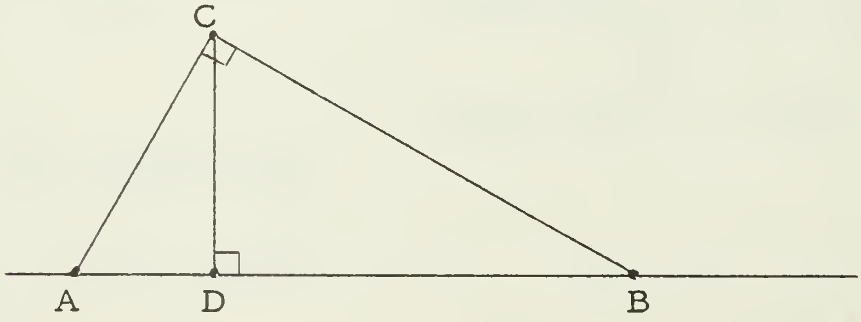
2. (a) For the figure given in Exercise 1, name a triangle one of whose angles is $\angle A$, and name a triangle one of whose angles is $\angle D$.

   (b) Give two matchings of the vertices of these triangles with respect to which $\angle A$ and $\angle D$ are corresponding angles.

3. (a) For the figure given in Exercise 1, name two triangles each of which has $\angle A$ as one of its angles.

   (b) Now, give a matching of the vertices of these triangles with respect to which $\angle A$ and $\angle A$ are <u>not</u> corresponding angles.

4.



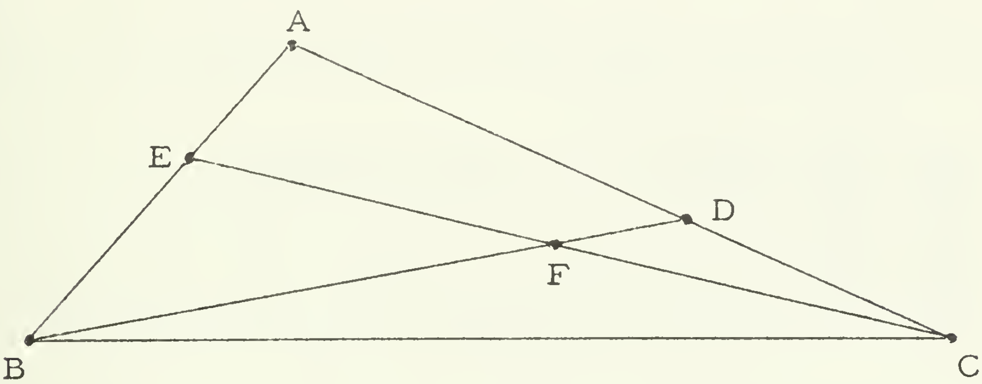
   (a) Give a matching of the vertices of $\triangle ACD$ with those of $\triangle CDB$ for which $\overleftrightarrow{CD}$ and $\overleftrightarrow{CD}$ are corresponding parts.

(b) Give a matching of the vertices of $\triangle ACD$ with those of $\triangle ACB$ for which $\overline{AD}$ and $\overline{AC}$ are corresponding sides.

(c) Give a matching for $\triangle ABC$ and $\triangle BCD$ for which $\angle B$ and $\angle B$ are corresponding angles.

(d) Give a matching for $\triangle ABC$ and $\triangle ACD$ for which the right angles are corresponding angles.

5.



(a) Name two triangles, and give a matching of their vertices with respect to which $\overline{EB}$ and $\overline{DC}$ are corresponding sides. [How many such pairs of triangles are there?]

(b) Name two triangles, and give a matching of their vertices with respect to which $\overline{BD}$ and $\overline{EC}$ are corresponding sides.

6. Each of the following exercises gives pairs of parts for two triangles. Tell the matching for which these parts are corresponding parts, and make a sketch of the triangles.

(a) $\overline{AB}$ and $\overline{MN}$, $\overline{AC}$ and $\overline{MQ}$, $\angle CAB$ and $\angle NMQ$.

(b) $\angle R$ and $\angle KPS$, $\angle TJR$ and $\angle S$, $\overline{PK}$ and $\overline{TR}$

(c) $\overline{AC}$ and $\overline{CD}$, $\overline{CB}$ and $\overline{CA}$, $\angle BCA$ and $\angle DCA$

(d) $\angle J$ and $\angle R$, $\overline{SJ}$ and $\overline{RP}$, $\angle JTS$ and $\angle J$

(e) $\overline{PB}$ and $\overline{BT}$, $\overline{MP}$ and $\overline{TM}$, $\overline{BM}$ and $\overline{MB}$

(f) $\overline{BP}$ and $\overline{MT}$, $\overline{PM}$ and $\overline{BT}$, $\angle PMB$ and $\angle MBT$

Page 6-88.

In each of the following exercises you are given certain informatio[n] which you may use along with other things to deduce that two of the triangles pictured are congruent. In order to reach this conclusion, you must show that there is a matching of the vertices which is a congruence. To do so, you can apply either the s.s.s. theorem or the s.a.s. theorem.

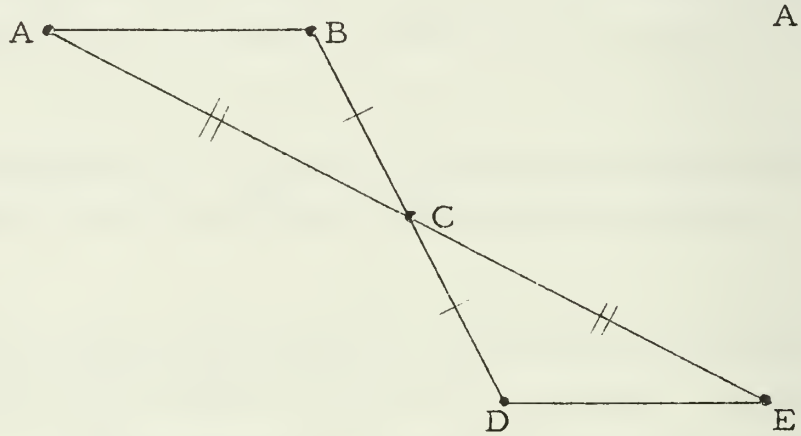
(a) Indicate the three pairs of congruent parts you would use to apply the theorem.

(b) Tell which theorem you are applying.

(c) Tell which matching is a congruence.

(d) Indicate the remaining pairs of congruent parts.

Sample. Given: $BC = CD$, $AC = CE$



Solution. [We notice immediately that $\angle ACB \cong \angle DCE$ because they are vertical angles. So, it looks as if the matching to be used is one for which $\overline{BC}$ and $\overline{CD}$, $\angle ACB$ and $\angle DCE$, and $\overline{AC}$ and $\overline{CE}$ are corresponding parts. Then, the s.a.s. theorem tells us that the matching is a congruence.]

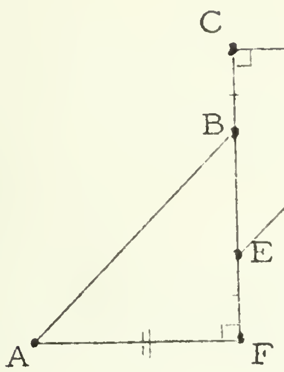
(a) $BC = CD$, $\angle ACB \cong \angle DCE$, $AC = CE$

(b) s.a.s.

(c) $BCA \longleftrightarrow DCE$ is a congruence

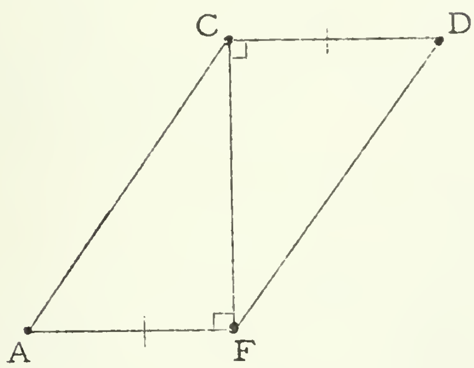
(d) $\angle B \cong \angle D$, $BA = DE$, $\angle A \cong \angle E$

1.



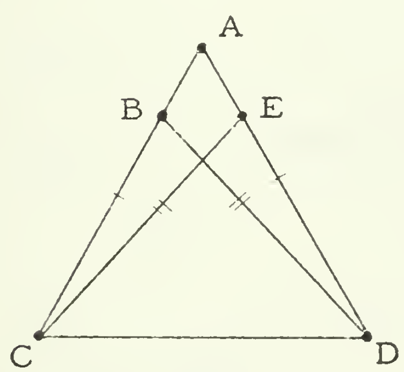
Given: $BC = EF$,
$CD = AF$,
$\overleftrightarrow{FC} \perp \overleftrightarrow{CD}$,
$\overleftrightarrow{CF} \perp \overleftrightarrow{FA}$

2.



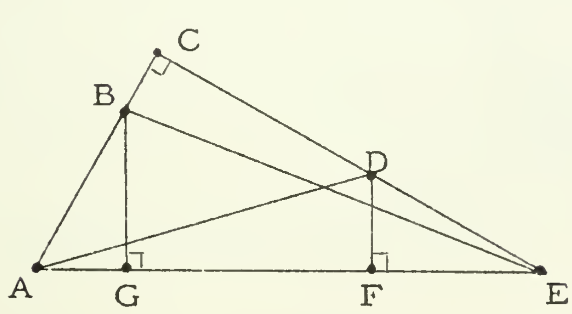
Given: $CD = AF$,
$\overleftrightarrow{FC} \perp \overleftrightarrow{CD}$,
$\overleftrightarrow{CF} \perp \overleftrightarrow{FA}$

3.



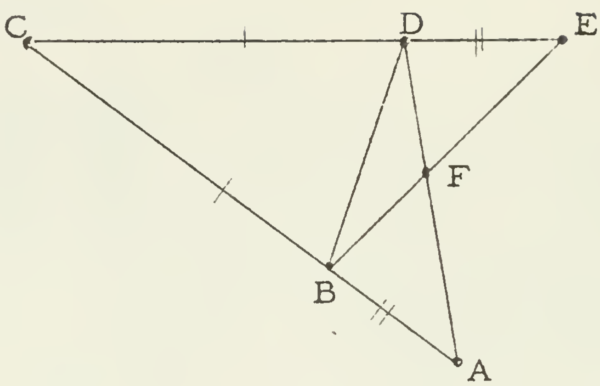
Given: $CE = BD$,
$BC = DE$

4.



Given: $\angle BGF$, $\angle C$, $\angle DFG$ are right angles,
$AG = DF$,
$BG = FE$

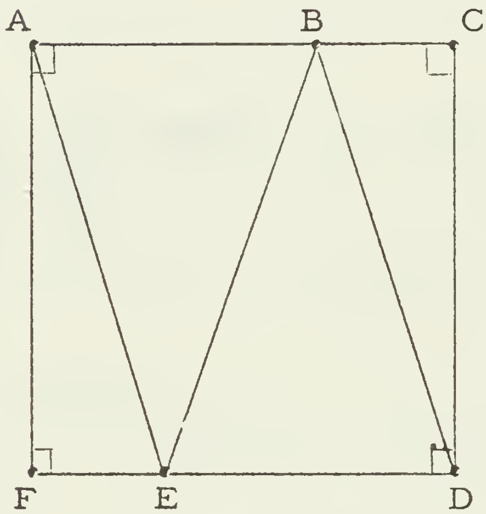
5.



Given: CB = CD

BA = DE

6.

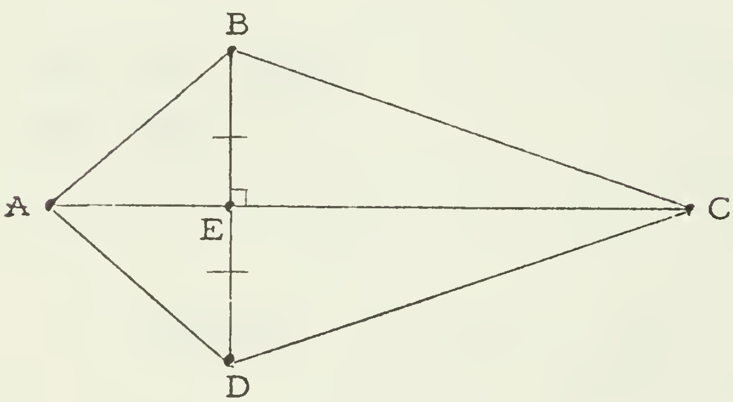


Given: $\angle$FAC, $\angle$C, $\angle$CDF, and $\angle$F are right angles,

AC = CD = DF = FA,

BC = FE

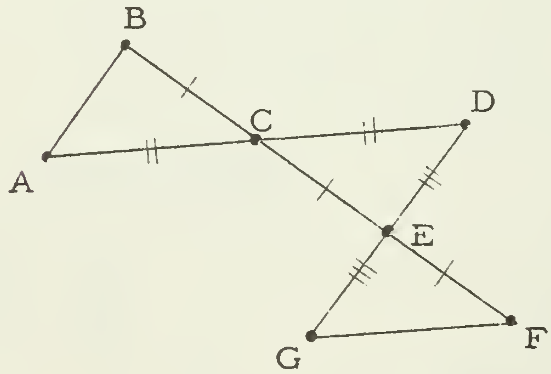
7.



Given: $\overleftrightarrow{AC} \perp \overleftrightarrow{BD}$ at E,

BE = ED

8.



Given: BC = CE = EF,

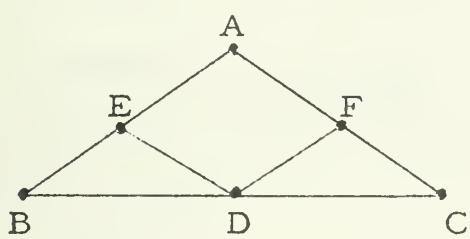
AC = CD,

GE = ED

Page 6-90.

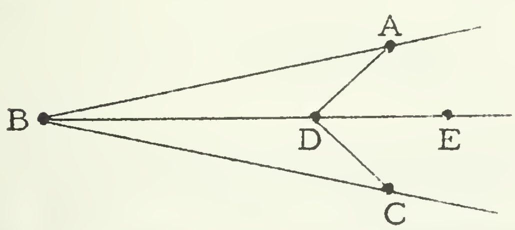
1. Given the triangles $\triangle RST$ and $\triangle MRQ$. If $RS = 5$, $TS = 6$, $MQ = 6$, $RM = 5$, and $m(\angle S) = 70 = m(\angle M)$, show that the given triangles are congruent.

2. If, in $\triangle ABC$ and $\triangle MPQ$, $\angle B$ and $\angle P$ are right angles, $MP = AB$, and $BC = PQ$, show that $AC = MQ$.

3. 

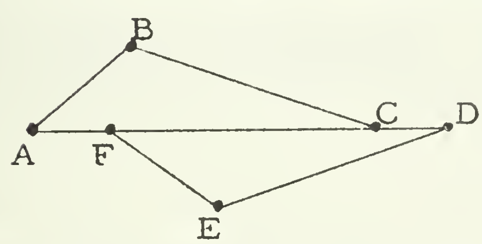
Hypothesis: $\angle C \cong \angle B$,
$FC = EB$,
D is the midpoint of $\overleftrightarrow{BC}$

Conclusion: $\angle FDC \cong \angle EDB$

4. 

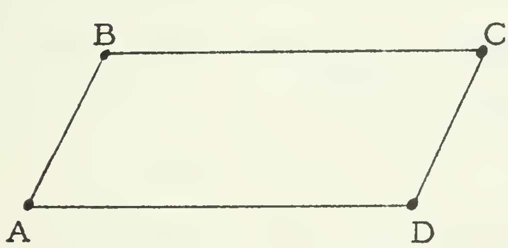
Hypothesis: $\angle ADE \cong \angle CDE$,
$D \in \overline{BE}$,
$AD = DC$

Conclusion: $\angle ABE \cong \angle CBE$

5. 

Hypothesis: $AB = FE$,
$BC = ED$,
$AF = CD$

Conclusion: $\angle B \cong \angle E$

6. 

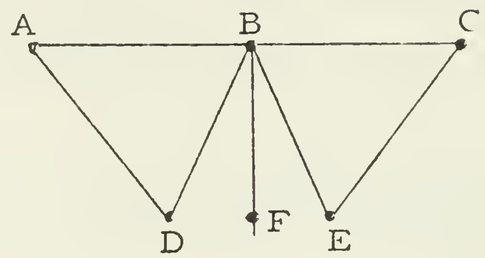
Hypothesis: $\overleftrightarrow{AB} \cong \overleftrightarrow{DC}$,
$\overleftrightarrow{BC} \cong \overleftrightarrow{DA}$

Conclusion: $\angle B \cong \angle D$, $\angle A \cong \angle C$

[Hint. Consider $\triangle ABC$ and $\triangle CDA$.]

Page 6-95.

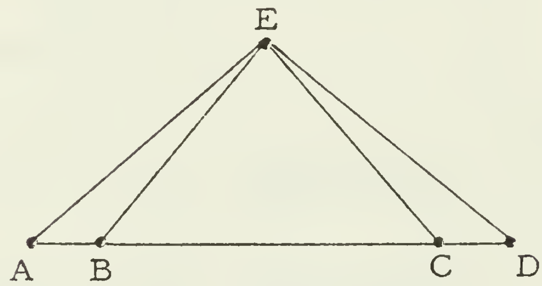
1.



Hypothesis: $\overleftrightarrow{BF}$ is the perpendicular bisector of $\overline{AC}$,
$\angle DBF \cong \angle EBF$,
$DB = EB$

Conclusion: $\angle D \cong \angle E$

2.



Hypothesis: $\overline{BE} \perp \overline{ED}$,
$\overline{CE} \perp \overline{EA}$,
$BE = CE$,
$ED = EA$

Conclusion: $AC = BD$

3. Draw $\overline{AB}$ such that $AB = 3$. Draw $\overrightarrow{AC}$ such that $\angle BAC$ is an angle of $20°$. Then, on the same side of $\overleftrightarrow{AB}$ as C, draw $\overrightarrow{AD}$, $\overrightarrow{AE}$, and $\overrightarrow{AF}$ such that $\angle BAD$ is an angle of $50°$, $\angle BAE$ is an angle of $80°$, and $\angle BAF$ is an angle of $100°$. Suppose that P is the point of $\overrightarrow{AD}$ such that $BP = 4$ and Q is the point of $\overrightarrow{AF}$ such that $AQ = 3$. What is QP?

4. Suppose B and D are two points on the sides of $\angle A$ such that $AB = AD$, and that C is a point in the interior of $\angle A$ such that $\angle CBA$ and $\angle CDA$ are supplements of $\angle A$. If $E \in \overline{BC}$ and $F \in \overline{DC}$ and $BE = DF$, show that $\angle BAE \cong \angle DAF$.

5. Draw $\triangle ABC$ such that $\angle B$ is an angle of $60°$. Locate D, the midpoint of $\overline{AB}$, and E, the midpoint of $\overline{BC}$. If $F \in \overline{AC}$, $EF = AD$, and $\angle FEC$ is an angle of $60°$, show that $\angle BDE \cong \angle EFC$.

Page 6-111.

1. For each of the sentences below, state the property which follows from it by definition.

   Sample 1. B is the midpoint of $\overline{AC}$

   Solution. $B \in \overline{AC}$ and $AB = BC$

   Sample 2. $\overrightarrow{AB} \cup \overrightarrow{AC}$ is an angle

   Solution. A, B, and C are noncollinear points

   (a) $\overrightarrow{BC}$ is the bisector of $\angle ABD$

   (b) $\overline{AB} \cong \overline{CD}$

   (c) $\angle A$ and $\angle B$ are complementary angles

   (d) $\angle K$ is a right angle

   (e) $\ell \perp m$

   (f) $\overleftrightarrow{AB}$ is the perpendicular bisector of $\overline{CD}$

   (g) $MTR \leftrightarrow SPQ$ is a congruence

   (h) $\triangle ABC$ is a scalene triangle

   (i) $\triangle ABC$ is an equiangular triangle

   (j) $\triangle ABC$ is an equilateral triangle

   (k) $\triangle ABC$ and $\triangle DEF$ are congruent

   (ℓ) $m(\angle E) + m(\angle D) = 180$

   (m) $\ell \cap m = \emptyset$

   (n) $\overleftrightarrow{AB} \perp \overline{CD}$ at P and P is the midpoint of $\overline{CD}$

   (o) A, B, and C are vertices of a triangle and ($AB = BC$ or $BC = CA$ or $CA = AB$)

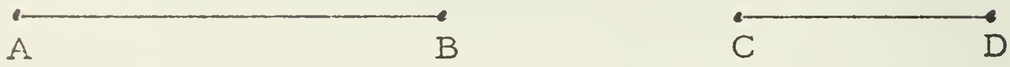
   (p) $\triangle MNR$ is an isosceles triangle with vertex angle $\angle R$

   (q) $\angle T$ is the supplement of $\angle T$

   (r) $\angle JKL$ is an acute angle

2. Use compass and straight-edge only to draw the figures which are described below.

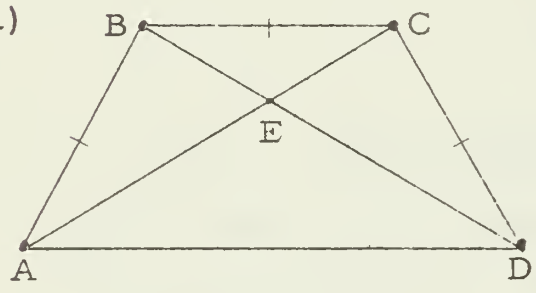
   (a) An isosceles triangle whose base is congruent to $\overline{AB}$ and one of whose legs is congruent to $\overline{CD}$.



   (b) An equilateral triangle one of whose sides is congruent to $\overline{AB}$ [see Exercise 2 (a)].

   (c) An isosceles triangle whose vertex angle is congruent to one of its base angles, and whose base is congruent to $\overline{AB}$ [see Exercise 2 (a)].

3. (a)



<u>Hypothesis:</u> $AB = BC = CD$, $\angle ABC \cong \angle BCD$

<u>Conclusion:</u> $AC = BD$

   (b) [Refer to Exercise 3 (a)]. Show that $\angle BAD \cong \angle CDA$.

   (c) Now, show that ΔEAD is isosceles.

4. (a) Suppose that A, B, and C are three noncollinear points. What is the total number of lines determined by pairs of these points?

   (b) Suppose that A, B, C, and D are four points. What is the total number of lines determined by pairs of these points?

   (c) Locate three points A, B, and C such that $\overrightarrow{BA} \cup \overrightarrow{BC}$ is neither a line nor an angle.

   (d) Locate three points A, B, and C such that

$$\overline{AB} \cup \{B\} \cup \overline{BC} \cup \{C\} \cup \overline{CA} \cup \{A\}$$

   is a triangle.

5. Suppose that G is a point in the interior of ΔABC such that $\overrightarrow{AG}$, $\overrightarrow{BG}$, and $\overrightarrow{CG}$ are the bisectors of $\angle A$, $\angle B$, and $\angle C$, respectively. Suppose further, that ΔABC is equilateral. Make a guess about the three triangles, ΔGAB, ΔGBC, and ΔGCA, and prove your guess.

6. (a) Suppose that A, B, and C are noncollinear points, and that $AB \not\cong AC$. Show why it does not follow that ΔABC is not isosceles.

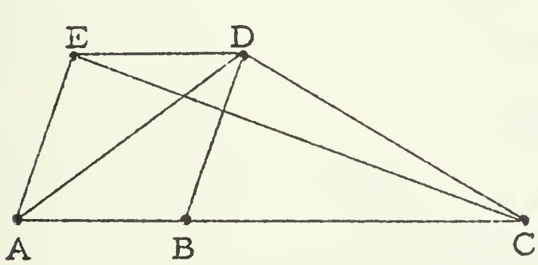
   (b) Suppose that, in ΔABC, $AB \not\cong BC$. Show that $\angle C$ is not congruent to $\angle A$.

   (c) Show that two sides of a triangle are not congruent if and only if the angles opposite these sides are not congruent.

7. Draw pictures to illustrate these situations.

   (a) Two sides and an angle of ΔABC are congruent, respectively, to two sides and an angle of $\Delta A'B'C'$, and ΔABC is not congruent to $\Delta A'B'C'$.

   (b) Three sides of ΔABC are congruent, respectively, to three sides of $\Delta A'B'C'$ and $ABC \leftrightarrow A'B'C'$ is not a congruence.

8. 

<u>Hypothesis:</u> $\overrightarrow{AD}$ is the bisector of $\angle A$,
$AE = AB = BD$

<u>Conclusion:</u> $\angle EAB \cong \angle EDB$

9. (a) Use your compass and straight-edge to draw an angle of $45°$.

   (b) Next, draw an angle of $22\frac{1}{2}°$.

   (c) Next, draw an angle of $67\frac{1}{2}°$.

Page 6-112.

1. Solve each of the following inequations.

 Sample. $3x + 2 > 14$

 Solution.
$$\begin{aligned} 3x + 2 &> 14 \\ 3x + 2 - 2 &> 14 - 2 \\ 3x &> 12 \\ 3x \div 3 &> 12 \div 3 \\ x &> 4 \end{aligned}$$

 Answer. $\{x\colon\ x > 4\}$

 (a) $x - 5 > 12$ (b) $x + 4 + 3x < 8$ (c) $3 + x > 3$

 (d) $3x > 2x + 1$ (e) $x < 8 - x$ (f) $3x + 1 < 2x + 4$

2. True or false ? [Give counter-examples for the false ones.]

 (a) $\forall_x \forall_y \forall_z$ $x > y$ if and only if $x + z > y + z$

 (b) $\forall_x \forall_y \forall_z$ if $x > y + z$ then $x > y$

 (c) $\forall_x \forall_y \forall_{z > 0}$ if $x > y + z$ then $x > y$

 (d) $\forall_x \forall_y$ $x + y > x$

 (e) $\forall_x \forall_{y > 0}$ $x + y > z$

 (f) $\forall_x \forall_y \forall_u \forall_v$ if $x > y$ and $u > v$ then $x + u > y + v$

 (g) $\forall_x \forall_y \forall_z$ if $x > y$ and $z > y$ then $x > z$

 (h) $\forall_x \forall_y \forall_z$ if $x > y$ and $y > z$ then $x > z$

3. Suppose that $AB = A'B'$ and that $BC = B'C'$.

 (a) Does it follow that $AB + BC = A'B' + B'C'$?

 (b) Does it follow that $AC = A'C'$?

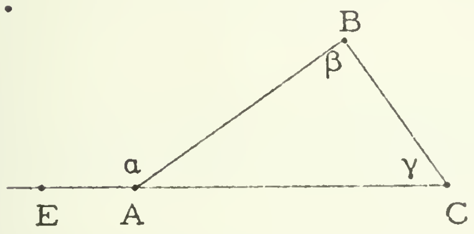
4. (a) If $AB > A'B'$, does it follow that $AB \neq A'B'$?

 (b) If $AB > A'B'$, does it follow that $AB \not< A'B'$?

 (c) If $AB \not< A'B'$, does it follow that $AB > A'B'$?

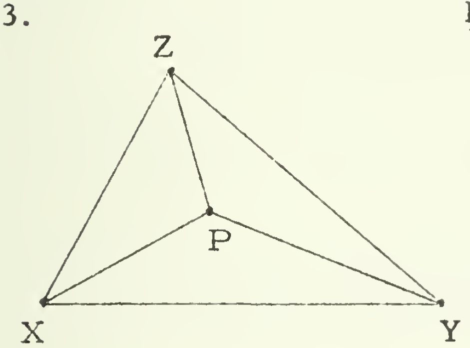
Page 6-138.

1. Prove that the altitude from a vertex of a triangle is not longer than the angle bisector or the median from that vertex.

2. 

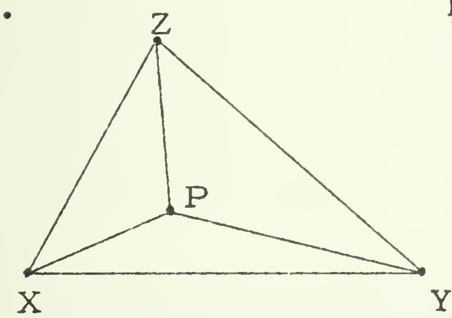
$\alpha = 140$, $\beta = 60$ $\gamma = 80$

Which is the longest side of $\triangle ABC$?

3. 

Hypothesis: $P \in$ interior of $\triangle XYZ$

Conclusion: $PX + PY + PZ > \frac{1}{2}(XY + YZ + ZX)$

4. 

Hypothesis: $P \in$ interior of $\triangle XYZ$

Conclusion: $PX + PY + PZ < XY + YZ + ZX$
[See Exercise 3 on 6-113.]

5. Prove that the measure of a median of a triangle is less than half the perimeter.

6. Draw $\triangle ABC$. Bisect an exterior angle at A and an exterior angle at B. Bisect $\angle C$. Prove that the lines containing these three angle bisectors are concurrent.

<u>Page 6-151.</u>

1. Suppose the measures of the three angles of a triangle are $2x$, $x + 30$, and $3x$. The measure of the largest angle of the triangle is ______.

2. The measure of the vertex angle of an isosceles triangle is 70. Find the measure of a base angle.

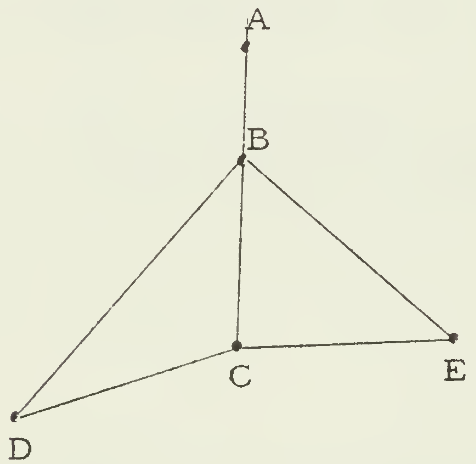
3. Two angles are supplementary, and the measure of one is 32 more than the measure of the other. Find the measure of the smaller.

4. Consider two parallel lines and a transversal. The measure of one of the two consecutive interior angles is 30 more than the measure of the other. What is the measure of the smaller angle?

5. If two lines are parallel, the lines which contain the bisectors of a pair of consecutive interior angles are _____ to each other.

6. If two lines are parallel, the lines containing the bisectors of a pair of corresponding angles are ______ to each other.

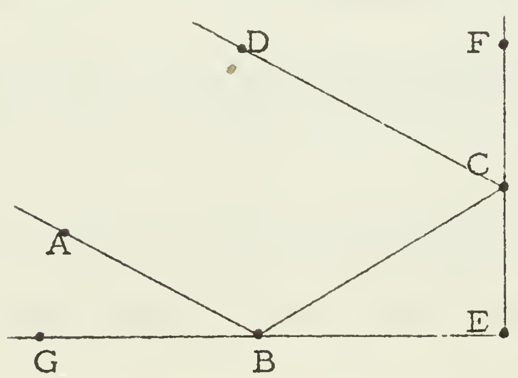
7.



<u>Hypothesis:</u> A, B, and C are collinear, $\angle ABD \cong \angle ABE$

<u>Conclusion:</u> $\overrightarrow{BC}$ bisects $\angle DBE$

8.



<u>Hypothesis:</u> $\angle ABG \cong \angle CBE$, $\angle BCE \cong \angle DCF$, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

<u>Conclusion:</u> $\angle CEB \cong$ _____

9. In $\triangle ABC$, $\angle C$ is an angle of $60°$ and $\angle A$ is an angle of $65°$. Which side of the triangle is the longest?

10. In $\triangle DEF$, $\angle D$ is obtuse and $\angle E$ is an angle of $45°$. What is the shortest side?

11. If a leg of an isosceles triangle is longer than the base, what can you say about the measure of the angle opposite the base?

12. For each three lines m, n, and p, if $m \parallel n$ and $p \perp m$ then p ____________________.

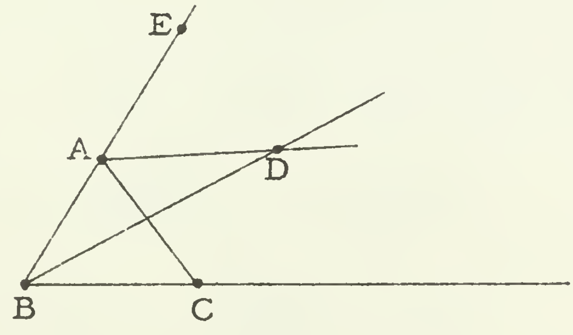
13. For each three lines m, n, and p, if $m \parallel n$ and $p \parallel m$ then p ____________________.

14. [True? False?] If a line is parallel to one of a pair of parallel lines, it is parallel to the other.

15. The measure of one of the acute angles of an isosceles right triangle is ________.

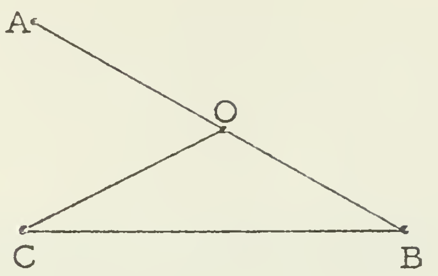
16. Suppose B and C are points one on each side of $\angle A$ such that $AB = AC$. Let E, D, and F be points in the interior of $\angle A$ such that $E \in \overline{BD}$ and $F \in \overline{CD}$. If $\angle EAB \cong \angle FAC$ and $\angle EAD \cong \angle DAF$, show that $\angle AED \cong \angle AFD$.

17. Suppose that $\overleftrightarrow{BD}$ bisects $\angle ABC$, $\overleftrightarrow{AD}$ bisects $\angle CAE$, $m(\angle ABC) = 60$, and $m(\angle ACB) = 50$. Find the measure of $\angle BDA$.



Page 6-157.

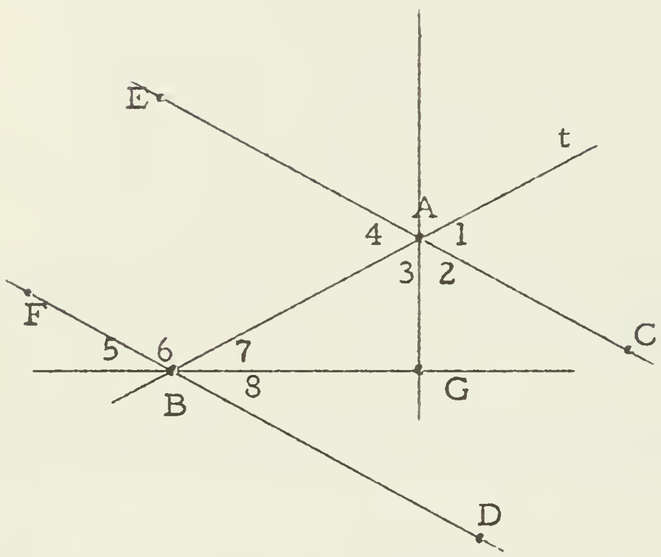
1.



Hypothesis: O is the midpoint of $\overline{AB}$, AO = CO

Conclusion: $m(\angle OBC) = \frac{1}{2} \cdot m(\angle AOC)$

2.



Hypothesis: $\overleftrightarrow{EC} \parallel \overleftrightarrow{FD}$,
$t \cap \overleftrightarrow{EC} = \{A\}$,
$t \cap \overleftrightarrow{FD} = \{B\}$,
$\overrightarrow{AG}$ bisects $\angle CAB$,
$\overrightarrow{BG}$ bisects $\angle ABD$

Conclusion: $m(\angle AGB) = 90$

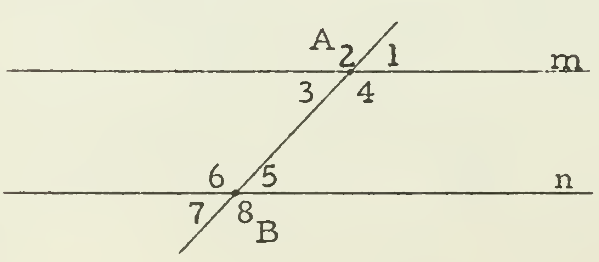
3. Draw isosceles triangle $\triangle DEF$ with vertex angle $\angle E$. If $A \in \overrightarrow{DE}$, $B \in \overline{EF}$, and $m(\angle EAB) = m(\angle EBA)$, prove that $\overleftrightarrow{AB} \parallel \overleftrightarrow{DF}$.

4. If $m \parallel n$ then

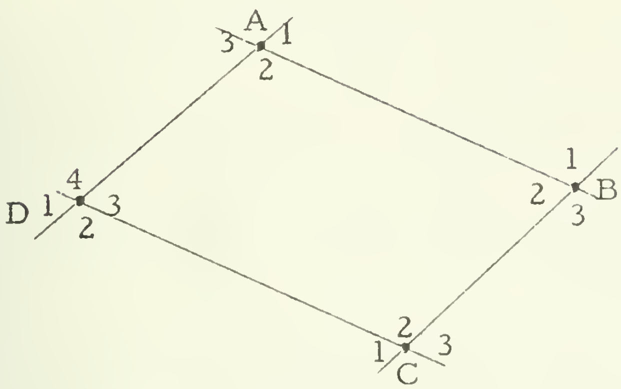
(a) if $m(\angle A_1) = 70$ then $m(\angle B_7) =$ ________.

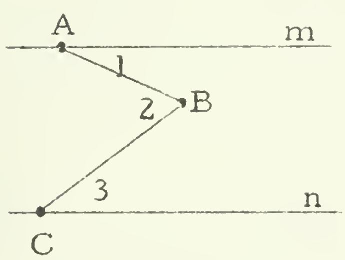
(b) if $m(\angle A_3) = 50$ then $m(\angle B_6) =$ ________.

(c) if $m(\angle B_8) = 100$ then $m(\angle A_2) =$ ________.



5. If $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$, and $m(\angle A_1) = 65$, find the measures of $\angle B_2$, $\angle C_2$, $\angle C_3$, and $\angle D_3$.



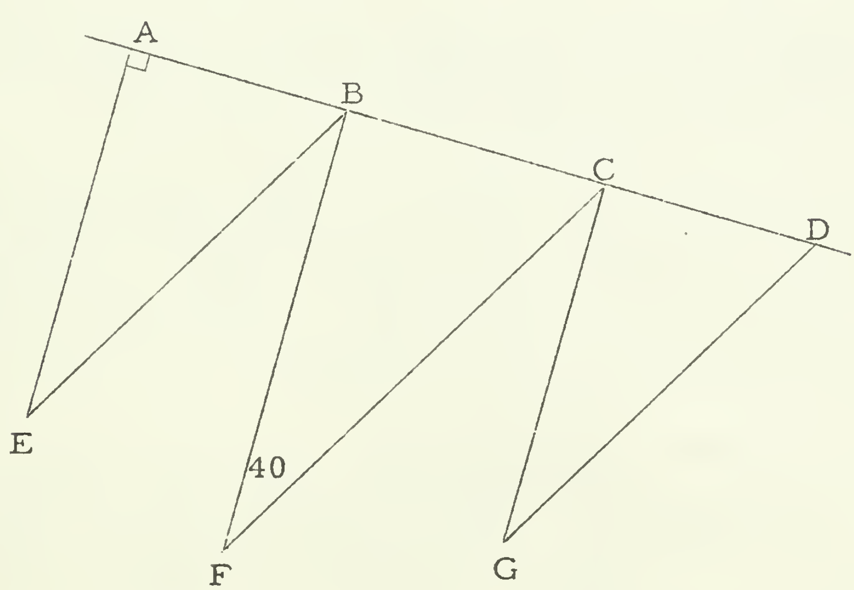
6. 

Hypothesis: $m \parallel n$

Conclusion: $m(\angle B_2) = m(\angle A_1) + m(\angle C_3)$

7. Draw a line $\overleftrightarrow{MN}$ through A of ΔABC parallel to $\overleftrightarrow{BC}$ such that $A \in \overline{MN}$. If $\angle MAB \cong \angle NAC$, prove that ΔABC is isosceles.

8. Suppose that $\overleftrightarrow{AE} \parallel \overleftrightarrow{BF}$, $\overleftrightarrow{BF} \parallel \overleftrightarrow{CG}$, $\overleftrightarrow{EB} \parallel \overleftrightarrow{FC}$, $\overleftrightarrow{EB} \parallel \overleftrightarrow{GD}$, $\overleftrightarrow{AE} \perp \overleftrightarrow{AD}$, and $m(\angle F) = 40$.



Find $\angle D$, and justify your answer.

Page 6-178.

1. Suppose that, in ΔABC, $BC = 9x + 3$ and the measure of the segment whose end points are the midpoints of $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ is $5x - 1$. Find the value of 'x' which satisfies these conditions.

2. In rectangle ABCD, M is the midpoint of $\overleftrightarrow{AB}$ and N is the midpoint of $\overleftrightarrow{BC}$. If $MN = 15$, how long is $\overleftrightarrow{BD}$?

3. Suppose that in parallelogram ABCD, $AB = x + 4$, $DC = 3x - 36$, and $BC = 2x - 16$. How long is $\overleftrightarrow{DA}$? Show that ABCD is a rhombus.

4. Suppose that ΔMNP is a right triangle with the right angle at M. R and S are midpoints of $\overleftrightarrow{MN}$ and $\overleftrightarrow{MP}$, respectively, and $m(\angle N) = 30$. If $RS = 10$, how long is $\overleftrightarrow{MP}$?

5. Suppose that $\overleftrightarrow{BT}$ is the angle bisector of ΔABC which is right-angled at B. If $BT = 10$ and $BA = BC$, how long is the hypotenuse of ΔABC?

6. The measure of one base of a trapezoid is 17 and the measure of the other base is 22. How long is the median of the trapezoid?

7. Prove that the diagonal of a rectangle is longer than each side.

☆ 8. Prove that the sum of the measures of two segments from a point of the base of an isosceles triangle to the feet of the perpendiculars through this point to the legs of the triangle is equal to the measure of the altitude to either leg. [Hint: Draw the segment through the point in the base perpendicular to the altitude.]

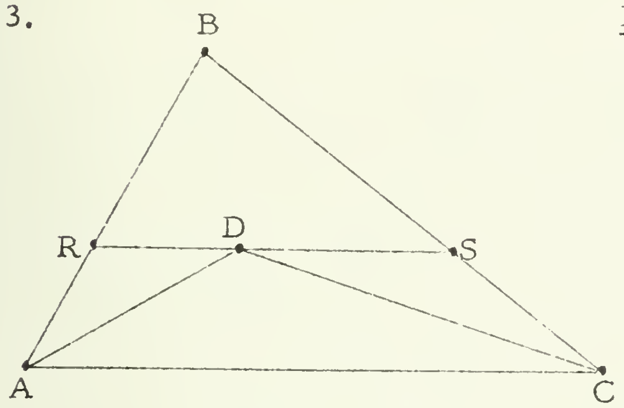
☆ 9. Prove that the sum of the measures of three segments from a point in the interior of an equilateral triangle to the feet of the perpendiculars through this point to the sides of the triangle is equal to the measure of an altitude of the triangle.

10. Suppose that ABCD is a parallelogram with $m(\angle A) = 40$, $AD = 6$, and $AB = 10$. If X and Z are on opposite sides of ABCD, $\overleftrightarrow{XZ}$ contains Y, the intersection of the diagonals of ABCD, $m(\angle BXZ) = 34$, and $XY = 3.5$, how long is $\overleftrightarrow{XZ}$?

11. The measure of a segment which joins the midpoints of two sides of an equilateral triangle is 10. What is the perimeter of the triangle?

12. The hypotenuse of a right triangle is twice as long as the shorter leg. What is the measure of the smallest angle of the triangle?

13.



Hypothesis: $\overleftrightarrow{RS} \parallel \overleftrightarrow{AC}$,
$\overrightarrow{AD}$ bisects $\angle A$,
$\overrightarrow{CD}$ bisects $\angle C$,
$AR = 18$, and $RS = 50$

Conclusion: CS = ____________

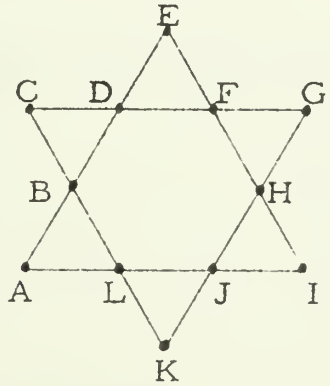
14. Suppose ABCD is a rhombus and $m(\angle B) = 120$. If $AB = 7$, find BD.

15. One of the base angles of an isosceles triangle is an angle of 15°. The side opposite this angle is 7 inches long. How far is the vertex of this angle from the opposite side? [It's not 7 inches!]

16. In parallelogram UVWX the measure of $\angle U$ is twice that of $\angle V$. What is the measure of $\angle W$?

17. Use your compass and straight-edge to draw an angle of 30°.

18.



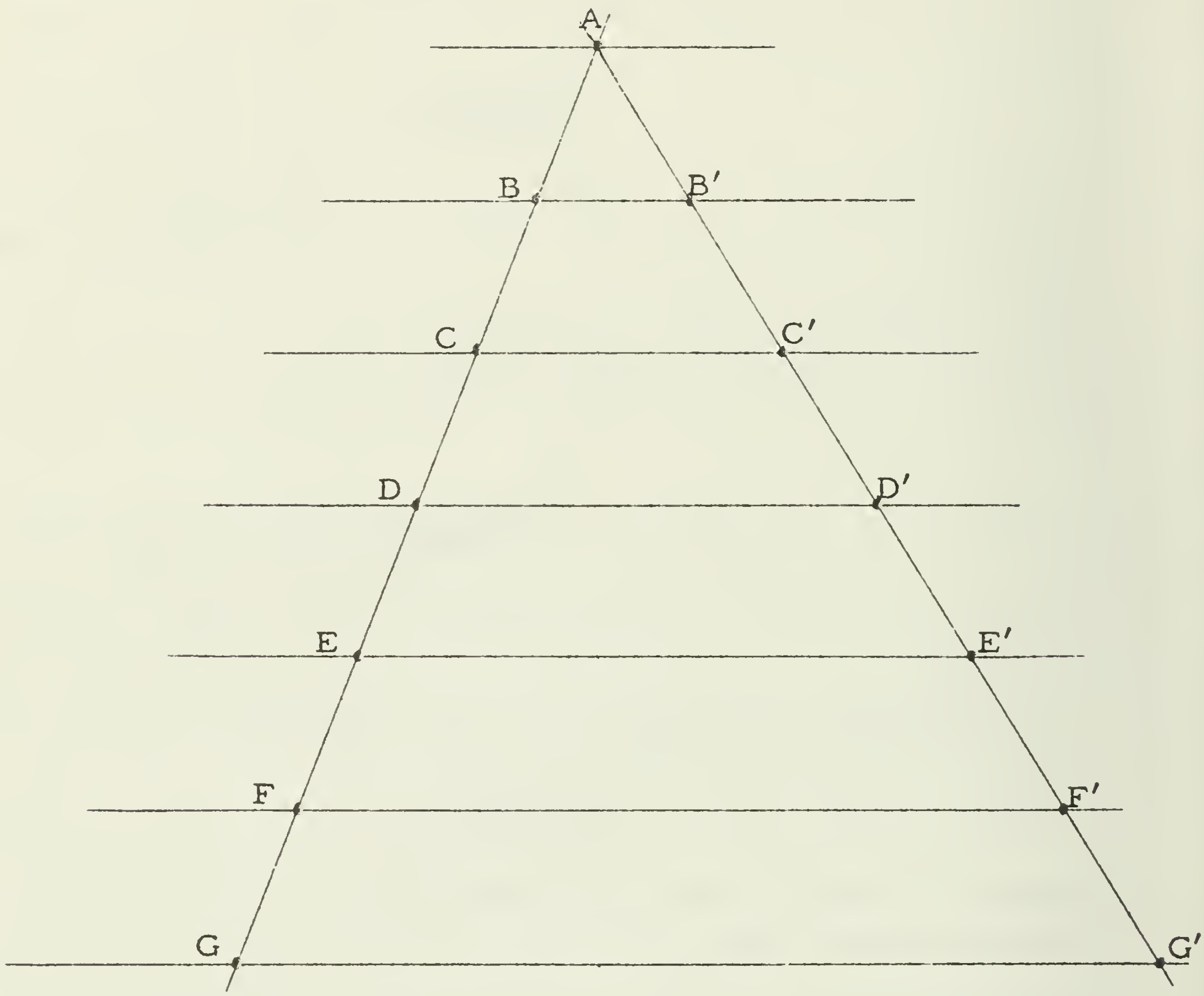
Hypothesis: BDFHJL is a regular hexagon

Conclusion: ΔAIE is equilateral

19. The figure ABCDEFGHIJKL shown in Exercise 18 may be called a regular six pointed star polygon, or, for short: a regular six pointed star. Draw a regular five pointed star and find the measures of all its angles.

Page 6-192.

1. Suppose that $\overleftrightarrow{AG}$ and $\overleftrightarrow{AG'}$ are transversals of the parallel lines as shown, and that AB = BC = CD = DE = EF = FG.



What theorem tells you that AB′ = B′C′ = C′D′ = D′E′ = E′F′ = F′G′?

2. Fill in the blanks.
   (a) AC/CF = 2/3; therefore, AC′/C′F′ = _______.
   (b) AD/DF = 3/2; therefore, AD′/AF′ = _______. [Careful!]
   (c) AC/DG = 2/3; therefore, AC′/D′G′ = _______.
   (d) AE/CG = _____; therefore, G′C′/E′A = ________.

3. (a) If AC = 7 then CF = _______ and A′C′/C′F′ = ________.
   (b) If AC = $\sqrt{2}$ then CG = _______ and A′C′/C′G′ = _______.
   (c) If AC = 14 and B′C′ = 9 then AG/AG′ = _______.

Page 6-202.

1. How many numbers are there whose square is 25?
2. How many <u>positive</u> numbers are there whose square is 144?
3. How many positive numbers are there whose square is 701?
4. $\forall_{p \geq 0}\ \sqrt{p} \geq$ ________
5. $\sqrt{25} \times \sqrt{25} =$ ________
6. $\sqrt{9} \times \sqrt{9} =$ ________
7. $\sqrt{144} \times \sqrt{144} =$ ________
8. $\sqrt{81} \times \sqrt{81} =$ ________
9. $(\sqrt{36})^2 =$ ________
10. $\sqrt{30} \times \sqrt{30} =$ ________
11. $\sqrt{2} \times \sqrt{2} =$ ________
12. $\sqrt{783} \times \sqrt{783} =$ ________
13. $\sqrt{5163} \times \sqrt{5163} =$ ________
14. $\sqrt{971 \times 835} \times \sqrt{971 \times 835} = 971 \times$ ________
15. $\sqrt{589 \times 762} \times \sqrt{589 \times 762} =$ ________ $\times 762$
16. $\sqrt{347 \times 698} \times \sqrt{698 \times 347} =$ ________ $\times$ ________
17. $\sqrt{1873 \times 6952} \times \sqrt{1873 \times 6952} =$ ________ $\times$ ________
18. $(\sqrt{1873} \times \sqrt{1873})(\sqrt{6952} \times \sqrt{6952}) =$ ________ $\times$ ________
19. $(\sqrt{1873} \times \sqrt{6952})(\sqrt{1873} \times \sqrt{6952}) =$ ________ $\times$ ________
20. $572 \times 189 = (\sqrt{572} \times \sqrt{572})(\sqrt{\_\_\_\_} \times \sqrt{\_\_\_\_})$
21. $64 \times 36 = (\sqrt{\_\_\_\_} \times \sqrt{\_\_\_\_})(\sqrt{64} \times \sqrt{36})$
22. $64 \times 36 = \sqrt{64 \times 36} \times \sqrt{\_\_\_\_ \times \_\_\_\_}$
23. $\sqrt{5 \times 7} \times \sqrt{5 \times 7} =$ _____ $\times$ _____
24. $(\sqrt{5} \times \sqrt{7}) \times (\sqrt{5} \times \sqrt{7}) =$ _____ $\times$ _____
25. How many positive numbers are there whose square is 5×7?
26. $\sqrt{5 \times 7} = \sqrt{\_\_\_\_} \times \sqrt{\_\_\_\_}$
27. $\sqrt{18} \times \sqrt{2} = \sqrt{\_\_\_\_ \times \_\_\_\_}$
28. $\forall_{p \geq 0}\ \sqrt{p}\sqrt{p} =$ ________
29. $\forall_{a \geq 0}\ \forall_{b \geq 0}\ (\sqrt{a}\sqrt{a})(\sqrt{b}\sqrt{b}) =$ ________
30. $\forall_{a \geq 0}\ \forall_{b \geq 0}\ (\sqrt{a}\sqrt{b})(\sqrt{a}\sqrt{b}) =$ ________

31. $\forall_{a \geq 0} \forall_{b \geq 0} \sqrt{a}\sqrt{b}$ is <u>the</u> positive number whose square is ______

32. $\forall_{a \geq 0} \forall_{b \geq 0} \sqrt{ab}\ \sqrt{ab} =$ ______

33. $\forall_{a \geq 0} \forall_{b \geq 0} \sqrt{\_\_\_\_\_}$ is <u>the</u> positive number whose square is ab

34. In view of the results in Exercises 31 and 33, if ab is nonnegative, how many positive numbers are there whose square is ab?
Hence, $\forall_{a \geq 0} \forall_{b \geq 0} \sqrt{a}\ \sqrt{b} =$ ______

35. $\sqrt{75} = \sqrt{\_\_\_\_\_ \times \_\_\_\_\_} = \sqrt{25} \times \sqrt{3} = \_\_\_\_\_ \times \sqrt{\_\_\_\_\_}$

36. $\sqrt{\_\_\_\_\_} = \sqrt{100 \times 5} = \sqrt{\_\_\_\_\_} \times \sqrt{5} = \_\_\_\_\_ \times \sqrt{5}$

37. $\sqrt{98} = \sqrt{\_\_\_\_\_ \times \_\_\_\_\_} = \sqrt{\_\_\_\_\_} \times \sqrt{\_\_\_\_\_} = \_\_\_\_\_ \times \sqrt{\_\_\_\_\_}$

38. $\sqrt{\_\_\_\_\_} = \sqrt{\_\_\_\_\_ \times \_\_\_\_\_} = \sqrt{\_\_\_\_\_} \times \sqrt{\_\_\_\_\_} = 2\sqrt{7}$

39. $\sqrt{\_\_\_\_\_} = \sqrt{\_\_\_\_\_ \times \_\_\_\_\_} = \sqrt{\_\_\_\_\_} \times \sqrt{\_\_\_\_\_} = 3\sqrt{11}$

40. $\sqrt{160} = \sqrt{\_\_\_\_\_ \times \_\_\_\_\_} = \sqrt{\_\_\_\_\_} \times \sqrt{\_\_\_\_\_} = \_\_\_\_\sqrt{10}$

41. $\sqrt{108} = \sqrt{\_\_\_\_\_ \times \_\_\_\_\_} = \sqrt{\_\_\_\_\_} \times \sqrt{\_\_\_\_\_} = 6\sqrt{\_\_\_\_\_}$

42. $\sqrt{63} = \sqrt{\_\_\_\_\_} \times \sqrt{\_\_\_\_\_} = \_\_\_\_\_ \times \sqrt{\_\_\_\_\_}$

43. $\sqrt{\_\_\_\_\_} = 11\sqrt{3}$ 44. $\sqrt{\_\_\_\_\_} = 3\sqrt{2}$

45. $\sqrt{2^2 + 6^2} = \sqrt{4 + 36} = \sqrt{4} \times \sqrt{\_\_\_\_\_} = \_\_\_\_\_$

46. Simplify each of the following.

(a) $\sqrt{9^2 + 15^2}$ [Ans: $9\sqrt{34}$] (b) $\sqrt{2^2 + 10^2}$ (c) $\sqrt{7^2 + 14^2}$

(d) $\sqrt{1^2 + 2^2}$ (e) $\sqrt{5^2 - 3^2}$ (f) $\sqrt{13^2 - 12^2}$ (g) $\sqrt{13^2 - 5^2}$

(h) $\sqrt{9^2 - 6^2}$ (i) $\sqrt{22^2 - 2^2}$ (j) $\sqrt{3^2 - 1^2}$ (k) $\sqrt{10^2 - 2^2}$

(ℓ) $\sqrt{s^2 + \frac{s^2}{4}}$ [Ans: $\left|\frac{s}{2}\sqrt{5}\right|$] (m) $\sqrt{x^2 + \frac{x^2}{9}}$ (n) $\sqrt{y^2 - \left(\frac{y}{2}\right)^2}$

(o) $\sqrt{t^2 + t^2}$ (p) $\sqrt{2k^2 - k^2}$ (q) $\sqrt{(2k)^2 - k^2}$

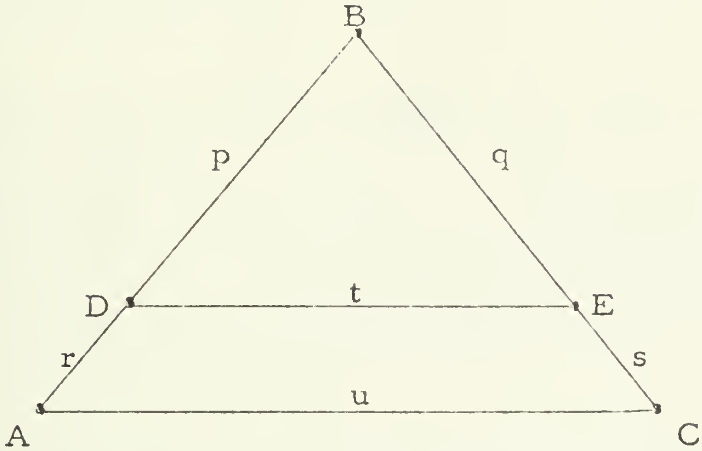
47. Simplify.

(a) $3 + 4 + \sqrt{3^2 + 4^2}$ (b) $5 + 12 + \sqrt{5^2 + 12^2}$ (c) $9 + 40 + \sqrt{9^2 + 40^2}$

(d) $(\sqrt{3})^2 + (\sqrt{5})^2$ (e) $(\sqrt{7})^2 + (\sqrt{9})^2 + (\sqrt{4})^2$ (f) $6^2 + (\sqrt{6})^2 + (2\sqrt{6})^2$

Page 6-219.

1.



If $\overleftrightarrow{DE} \parallel \overleftrightarrow{AC}$ then

(a) $\frac{p}{r} = \frac{q}{?}$ (b) $\frac{q+s}{q} = \frac{?}{p}$

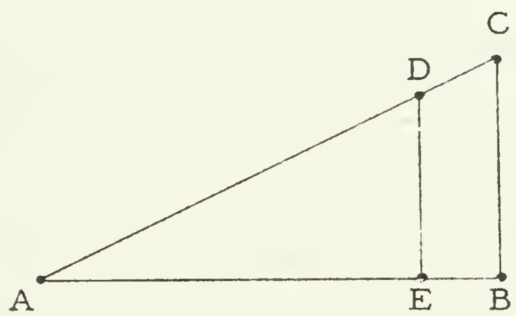
(c) $\frac{r}{s} = \frac{?}{?}$ (d) $\frac{q}{p} = \frac{?}{?}$

2. A line parallel to side $\overleftrightarrow{XY}$ of $\triangle XYZ$ intersects $\overleftrightarrow{XZ}$ in the point U and $\overleftrightarrow{YZ}$ in the point V. If UZ is 15, XU is 5, and VZ is 18, find YV.

3. Given $\triangle ABC$ with $D \in \overline{AB}$ and $E \in \overline{BC}$ such that $\overleftrightarrow{DE} \parallel \overleftrightarrow{AC}$. If BD = 8, DA = 6, and EC = 9, what is BE?

4. In $\triangle ABC$, $m(\angle A) = 90$ and $m(\angle B) = 40$. In $\triangle DEF$, $m(\angle D) = 40$ and $m(\angle E) = 50$. Is $\triangle ABC \sim \triangle DEF$?

5.



Given: $\overleftrightarrow{DE} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{CB} \perp \overleftrightarrow{AB}$, CB = 40, ED = 30, EB = 20

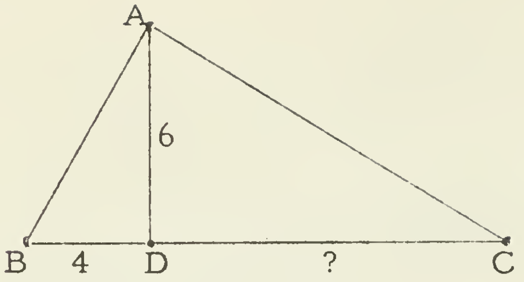
Find: AE

6. Draw an isosceles trapezoid with base-measures 5 and 10 and side-measure 4. How far must you extend each of the nonparallel sides to form a triangle?

7. $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$ at F. $\angle ACD \cong \angle DBA$ and $\angle CAB \cong \angle CDB$. AF = 8, FB = 3, and CF = 6. Find FD.

8. A line parallel to side $\overleftrightarrow{AC}$ of $\triangle ABC$ intersects $\overline{AB}$ in D and $\overline{CB}$ in E. If BE = 20, EC = 15, and BD = AD + 3, find AD.

9.



Given: $\triangle ABC$ is right-angled at A, $\overleftrightarrow{AD}$ is the altitude to the hypotenuse

Find: DC

10. The measure of the hypotenuse of a right triangle is 20, and the measure of the altitude to the hypotenuse is 6. Find the measures of the segments into which the foot of the altitude divides the hypotenuse.

11. The legs of a right triangle measure 2 and 5. What is the measure of the hypotenuse?

12. Find the measure of the altitude of an equilateral triangle whose side-measure is 8.

13. In right triangle $\triangle ABC$, $\overleftrightarrow{CD}$ is the altitude to the hypotenuse. If $AD = 6$ and $AB = 10$, what is AC?

14. What is the measure of a diagonal of a square whose side-measure is 15?

15. The hypotenuse of a right triangle is twice as long as the shorter leg. If the hypotenuse is 30 inches long, what is the length of the longer leg?

16. Suppose the measure of a side of a square is 3k. What is the measure of its diagonal?

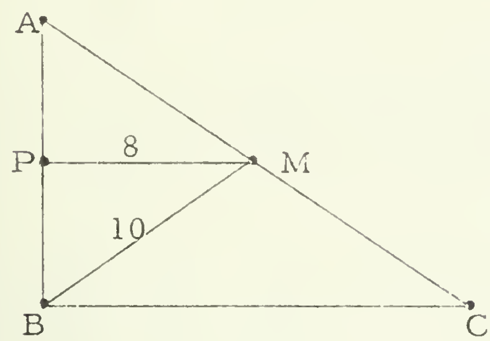
17. The measures of the diagonals of a rhombus are 12 and 16. Find the measure of a side of the rhombus.

18. If $AB = 1$ and P is a point such that $AP = BP = 2$, what is the distance between P and $\overleftrightarrow{AB}$?

19. The measures of the bases of an isosceles trapezoid are 9 and 15. If the measure of each of the nonparallel sides is 5, what is the distance between the bases?

20. In a right triangle, the projections of the two legs on the hypotenuse measure 18 and 32. What is the measure of the longer leg?

21. A doorway is 3 feet wide and 8 feet high. Can a circular table top 9 feet in diameter be carried through the doorway?

22. 

Given: $\angle B$ is a right angle, $\overline{BM}$ is a median, $\overleftrightarrow{MP} \perp \overleftrightarrow{AB}$, $BM = 10$, $MP = 8$

Find: perimeter of $\triangle ABC$

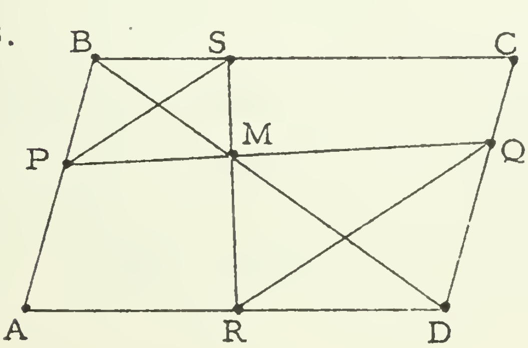
23. The lengths of two sides of a triangle are 10 inches and 14 inches, and the angle included between these sides is an angle of 30°. What is the length of the altitude to the 14-inch side?

24. The measure of one acute angle of a right triangle is twice the measure of the other. The longer leg measures $5\sqrt{3}$. What is the measure of the hypotenuse?

25. The measures of $\overline{AB}$ and $\overline{BC}$ of parallelogram ABCD are 6 and 16. If $m(\angle B)$ is 60, what is AC?

26. Find the lengths of the diagonals of a rhombus if each side is k inches long, and one of the angles of the rhombus is an angle of 60°.

27. What is the distance between the bases of an isosceles trapezoid if the measures of the bases are 8 and 14, and one of the base angles is an angle of 45°?

28. 

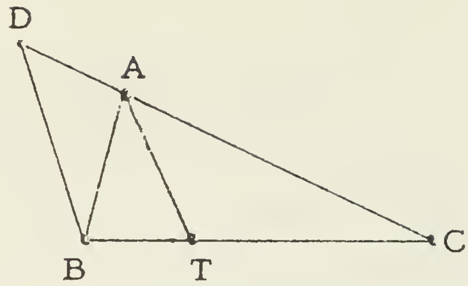
Hypothesis: ABCD is a parallelogram

Conclusion: $\frac{BM}{MD} = \frac{PM}{MQ}$,

$\frac{BM}{MD} = \frac{SM}{MR}$,

$\triangle PSM \sim \triangle QRM$

☆29. (a)

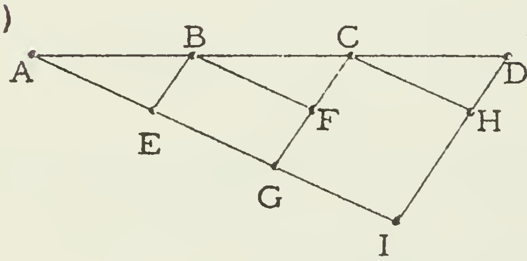


Hypothesis: $\overline{AT} \parallel \overline{DB}$,
$\overline{AT}$ is the angle bisector from A of $\triangle ABC$

Conclusion: $\dfrac{AC}{AB} = \dfrac{CT}{TB}$

(b) Given the 30-60-90 triangle $\triangle MPQ$ with $m(\angle M) = 30$ and $m(\angle P) = 60$. If $\overline{MN}$ is the angle bisector from M of $\triangle MPQ$, what is the ratio of $\overline{QN}$ to $\overline{NP}$?

☆30. (a)



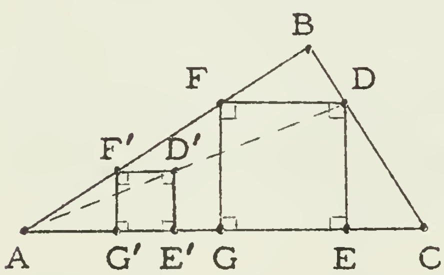
Hypothesis: $AE = EG$, $EG = GI$,
$\overleftrightarrow{BF} \parallel \overleftrightarrow{AI}$, $\overleftrightarrow{CH} \parallel \overleftrightarrow{AI}$,
$\overleftrightarrow{BE} \parallel \overleftrightarrow{CG}$, $\overleftrightarrow{BE} \parallel \overleftrightarrow{ID}$

Conclusion: $AB = BC$, $AB = CD$

[Hint. Use Exercise 6 on page 6-156.]

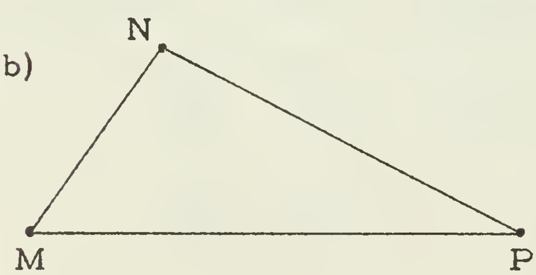
(b) Draw segments $\overline{AB}$ and $\overline{AC}$ such that $B \notin \overline{AC}$. Find the midpoint M of $\overline{AC}$. Draw a line ℓ parallel to $\overleftrightarrow{AB}$ through M. Let $\ell \cap \overleftrightarrow{CB} = \{D\}$. Draw a line m parallel to $\overleftrightarrow{AC}$ through D. Let $m \cap \overleftrightarrow{AB} = \{E\}$. What can you say about D and E?

☆31. (a)



The quadrilateral D′E′G′F′ is a square. The half-line $\overrightarrow{AD'}$ intersects side $\overline{BC}$ of $\triangle ABC$ in D. The quadrilateral DEGF is a rectangle. Show that it is also a square.

(b)



Using compass and straight-edge only, draw a square with one vertex in $\overline{MN}$, another in $\overline{NP}$, and a side contained in $\overline{MP}$.

Page 6-230.

1. Describe how a surveyor could use a transit, a table of trigonometric ratios, and a helper with a 50-foot chain to determine the width of a river.

2. (a) A forest ranger is stationed in a tower 75 feet tall. He spots smoke at a point whose angle of depression is an angle of 10°. How far from the base of the tower is the smoke?

   (b) If the bearing of the smoke from the tower is S63°E and the bearing of ranger headquarters, 2 miles from the tower, is S20°W, how far is the fire from ranger headquarters?

3. A pilot wishes to make a trip from Zabranchburg to Zilchville. Zilchville has a bearing of N30°E from Zabranchburg and is 230 "air-miles" away. His compass is faulty and he actually heads N31°E. If he continues on this heading, how close will he come to Zilchville?

4. A ladder leaning against a wall is in a safe position when the foot of the ladder is one foot from the base of the wall for each five feet of ladder. How far from the base of a wall should the foot of an 18 foot ladder be placed? What is the safe "angle of elevation" for a ladder which is 22.3 feet long? How far up a wall should the top of an 8 foot ladder be placed?

5. One of the longer diagonals of a regular hexagon is 8 inches long. What is the perimeter of the hexagon? What is the length of one of the shorter diagonals?

6. A forward observer is 1000 yards from a gun position and on top of a hill 6000 feet high. He sights a concentration of enemy tanks at a bearing of N70°W and an angle of depression of 25°. If his bearing on the gun position is S 10°E and the angle of depression of the gun position is an angle of 40°, what is the range and bearing of the tanks from the gun?

7. A pilot wants to take off from a field and clear a mountain range 10,000 feet high which is 15 miles from his take-off point. What is the minimum angle of climb?

<u>Page 6-316.</u>

1. What is the measure of a central angle which intercepts an arc of 37°?

2. If the measure of chord $\overline{AB}$ is the radius of the circle, what is the measure of the central angle which intercepts $\overset{\frown}{AB}$?

3. Find the measure of the arc intercepted by an inscribed angle of 70°.

4. $\overline{MN}$ is a diameter of a circle and $\overset{\frown}{MP}$ is an arc of 100°. What is $m(\angle PMN)$?

5. $\triangle ABC$ is inscribed in a circle. If the measures of $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ are 65 and 93, respectively, what is $m(\angle B)$?

6. If one of the congruent sides of an inscribed isosceles triangle has a minor arc of 100°, what is the measure of the vertex angle?

7. Suppose that the measures of the minor arcs determined by the legs of an inscribed right triangle are in the ratio 1 to 2. What is the measure of the smallest angle of the triangle?

8. Suppose that quadrilateral ABCD is inscribed in a circle. What is $m(\angle B)$ if $m(\angle D)$ is 63?

9. Chords $\overline{AB}$ and $\overline{CD}$ of a circle intersect at a point E. If the measures of $\overset{\frown}{AC}$ and $\overset{\frown}{BD}$ are 110 and 40, respectively, what is $m(\angle AEC)$? What is $m(\angle AED)$?

10. Suppose that ABCDEF is a regular hexagon inscribed in a circle, and that $\overleftrightarrow{PQ}$ is tangent to the circle at A such that $A \in \overline{PQ}$. What are $m(\angle PAC)$ and $m(\angle PAE)$? What is $m(\angle PAD)$?

11. Suppose that $\overleftrightarrow{PM}$ and $\overleftrightarrow{PN}$ are tangent to a circle and M and N are the points of tangency. If the measure of $\overset{\frown}{MN}$ is 50, what is $m(\angle MPN)$?

12. An angle which is a subset of the union of a secant and a tangent is an angle of 70°. If the larger intercepted arc is an arc of 165°, what is the measure of the smaller intercepted arc?

Page 6-334.

1. What is the measure of the angle between the tangents to a circle of radius 5 from a point which is at a distance 12 from the center?

2. Two chords intersect within a circle. The measures of the segments of one chord are 7 and 4. If the measure of a segment of the second chord is 12, find the measure of its other segment.

3. A tangent and a secant are drawn to a circle from an external point. The measure of the secant segment is 12, and that of its external segment is 3. Find the measure of the tangent segment.

4. A tangent and a secant are drawn to a circle from an external point. If the measure of the tangent segment is 12 and that of the external segment of the secant is 8, find the measure of the secant segment.

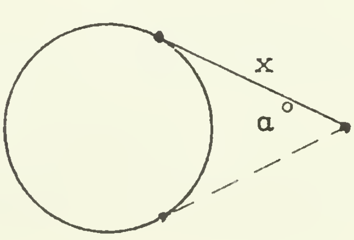
5. What is the radius of the circumcircle of a right triangle whose legs are of measures 9 and 12?

6. Prove that half the measure of a chord perpendicular to a diameter is the mean proportional between the measures of the segments into which the chord divides the diameter.

7. A 12-inch chord is 8 inches from the center of a circle. How long is a radius of the circle?

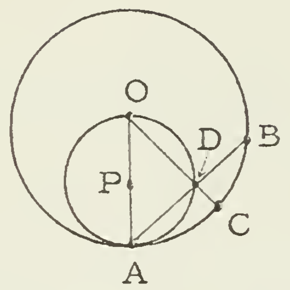
8. The diameters of two concentric circles are 10 and 26. Find the measure of a chord of the larger circle which is tangent to the smaller circle.

9. What is the length of a tangent segment from a point which is 8 inches from the center of a 4-inch circle? How many degrees are there in the angle formed by the two tangents from this point?



10. Suppose that O is the center of a circle and that A, B, and C are points of the circle. If $\overrightarrow{OB}$ bisects $\angle AOC$, show that $\overline{AB} \cong \overline{BC}$.

11.



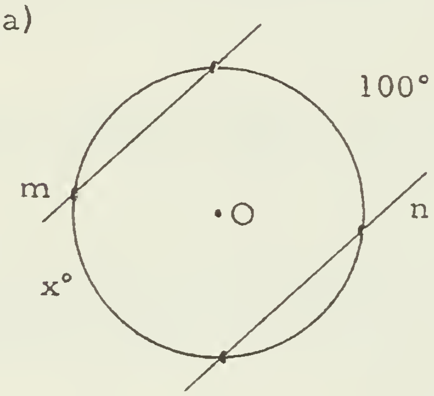
<u>Hypothesis:</u> O and P are centers, $\overline{OA}$ is a diameter

<u>Conclusion:</u> $\overset{\frown}{BC} \cong \overset{\frown}{AC}$

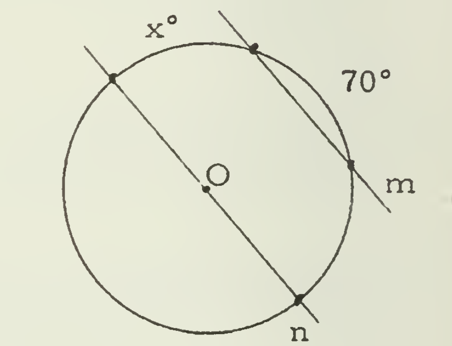
12. Prove that the bisector of an inscribed angle bisects the arc intercepted by the angle.

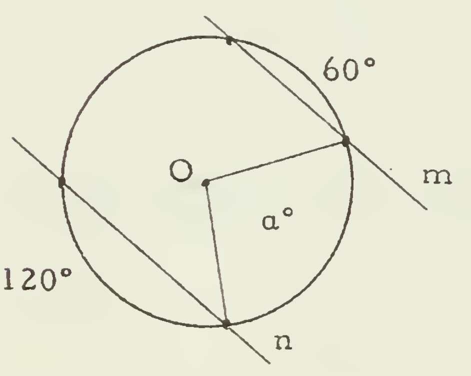
13. Find the indicated angle-or arc-measures. [In each case, m || n and O is the center.]

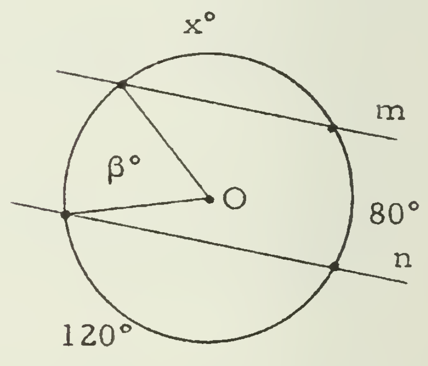
(a)



(b)





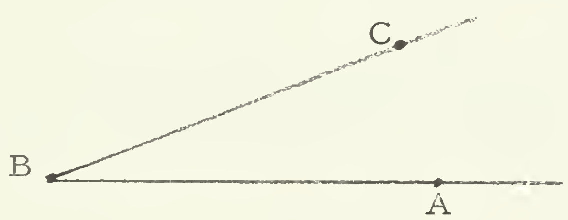


14. Suppose the points of tangency of three tangents to a circle, no two of which are parallel, are the vertices of an isosceles triangle. Prove that the points of intersection of the tangents [two at a time] are vertices of an isosceles triangle.

15. If chord $\overline{AB}$ of a circle is 15. 32 inches from the center and $\overset{\frown}{AB}$ is an arc of 80°, how long is a radius of the circle?

16. Suppose that the earth is a sphere and a rope is stretched tightly around the earth at the equator. The rope is then cut and a 40-foot length of rope is added. The new rope is then supported so that it and the equator are concentric circles. Could you put a 12-inch ruler between the equator and the rope? Could you crawl under the rope? Could you walk under the rope? Could you jump over the rope?

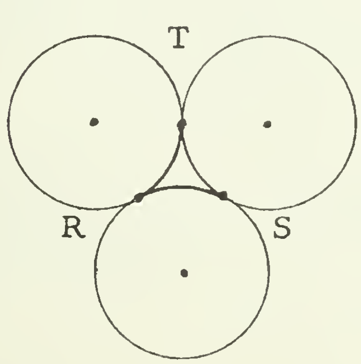
17. Use your compass and straight-edge and copy $\angle ABC$. Construct a circular arc of radius $\frac{1}{2}$ which is tangent to the sides of $\angle ABC$.



18. A running track consists of two parallel straightaways each a quarter of a mile long and two semicircular ends each a quarter of a mile long at the inner curb. Two men run once around this track, one 1 foot from the inner curb and the other 6 feet from the inner curb. How much farther does the second man run than the first?

19. The rim speed of a wheel is the distance through which a point on the rim passes in one minute. What is the rim speed, in feet, of the flywheel [radius: 1 foot] of an engine if the wheel revolves 250 times per minute?

20.

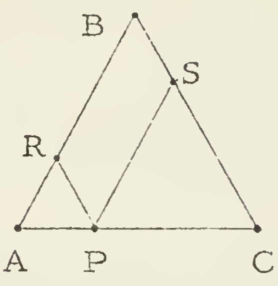


Given: three circles tangent as shown,
each circle has radius r

Find: $m(\widehat{SR}) + m(\widehat{RT}) + m(\widehat{TS})$

REVIEW EXERCISES

1. The perimeter of a rectangle is 200 and the ratio of a pair of adjacent sides is $\frac{3}{7}$. Find the area-measure of the rectangle.

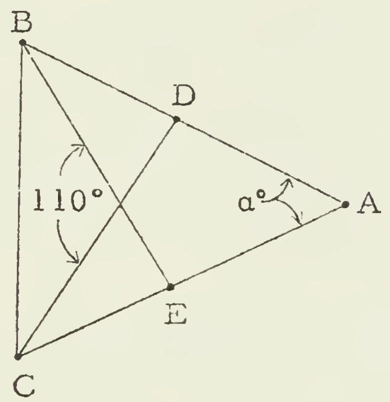
2. 

<u>Hypothesis:</u> $\overline{AB} \cong \overline{BC}$,
$\overleftrightarrow{PS} \parallel \overleftrightarrow{AB}$,
$\overleftrightarrow{PR} \parallel \overleftrightarrow{BC}$

<u>Conclusion:</u> $RP + PS = BC$

3. The point P divides the base $\overline{BC}$ of $\triangle ABC$ in the ratio $1 : 4$. Find the ratio of the area-measure of $\triangle ABP$ to that of $\triangle ACP$.

4. Prove that if the smaller base of an isosceles trapezoid is congruent with each of the legs, the diagonals bisect the angles adjacent to the longer base.

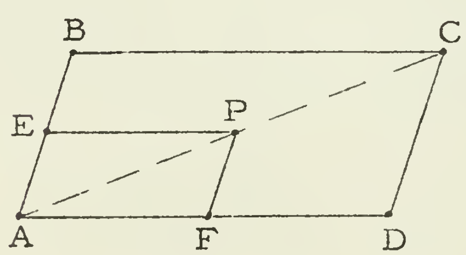
5. 

<u>Given:</u> $\overrightarrow{BE}$ and $\overrightarrow{CD}$ are angle bisectors,
$\overline{AB} \cong \overline{AC}$

<u>Find:</u> a

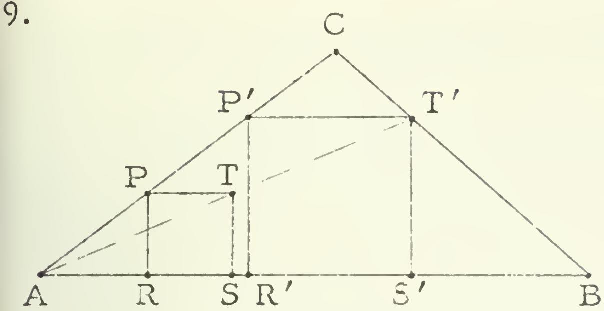
6. The area-measure of circle I is twice the area-measure of circle II. Find the ratio of radius I to radius II.

7. Show that the first and third quadrisection points of a diagonal of a parallelogram together with a pair of opposite vertices of the parallelogram are vertices of a second parallelogram.

8. 

<u>Hypothesis:</u> ABCD and AEPF are parallelograms

<u>Conclusion:</u> $ABCD \sim AEPF$

9. 

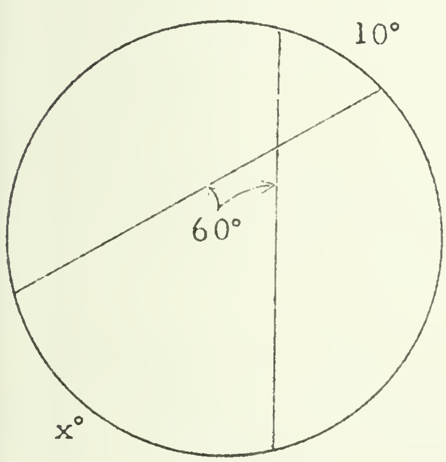
Hypothesis: PRST is a square,
A, T, T′ are collinear
$\overleftrightarrow{T'P'} \parallel \overleftrightarrow{BA}$, $\overleftrightarrow{T'S'} \perp \overleftrightarrow{AB}$,
$\overleftrightarrow{P'R'} \parallel \overleftrightarrow{T'S'}$.

Conclusion: P′T′S′R′ is a square

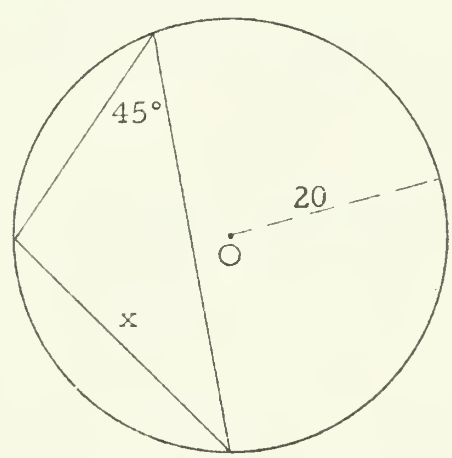
10. A see-saw is 10 feet long. If the fulcrum is 3 feet high and located at the middle of the board, how high above the level ground can one end of the board rise? Repeat the problem for a 15-foot see-saw.

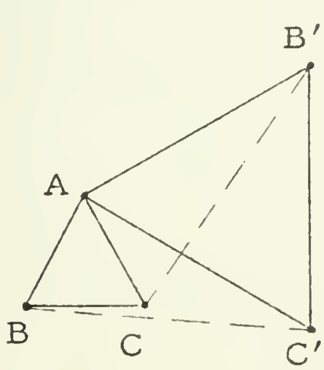
11. Two sides of a parallelogram measure 10 and 20, respectively. If the degree-measure θ of the included angle is a number between 10 and 80, what can you say about the area-measure? Repeat the problem for $10 \leq \theta \leq 100$.

12. Find x.



13. Find x.



14. 

Hypothesis: $\triangle ABC$ and $\triangle AB'C'$ are equilateral triangles

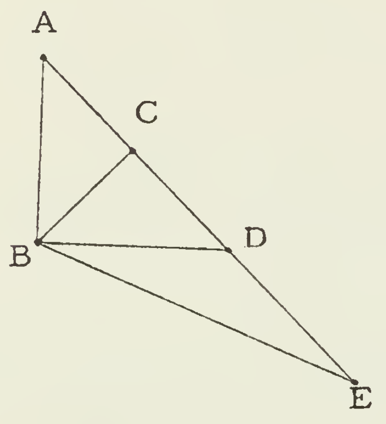
Conclusion: $\overline{BC'} \cong \overline{B'C}$

[Query. Do the triangles have to be equilateral in order to deduce the conclusion?]

15. For what acute angle is its sine ratio the same as its cosine ratio?

16. The sine ratio for an acute angle, $\angle A$, is $3/5$. Without using the table, compute the cosine and tangent ratios for this acute angle.

17.



Hypothesis: $AC = CB$, $CB = CD$, $BD = DE$

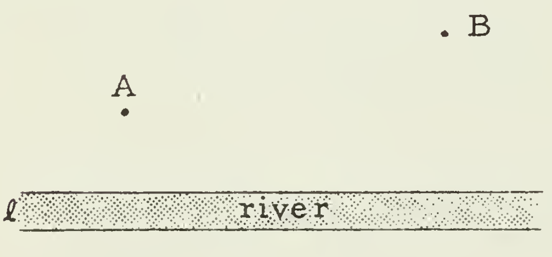
Conclusion: $m(\angle AEB) = \frac{1}{4}m(\angle BCA)$, $\angle A$ and $\angle ADB$ are complementary

18. Show that a triangle, two of whose vertices are the midpoints of the legs of a right triangle and whose third vertex is the foot of the altitude to the hypotenuse, is a right triangle.

19. A chord of the larger of two concentric circles is tangent to the smaller. If the chord measures 10, what is the area-measure of the circular ring?

20. The legs of a right triangle measure 7 and 24, respectively. Compute the three trigonometric ratios for each acute angle of the triangle, and compute the measures of these angles.

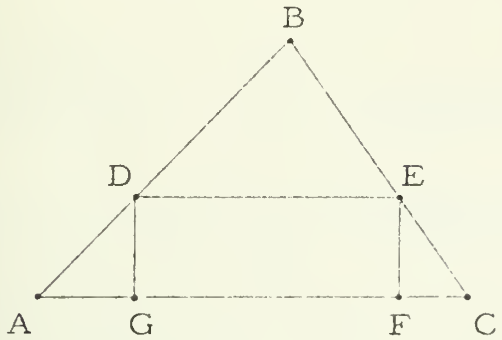
☆ 21.



A cowboy at A wants to ride to B as quickly as possible and water his horse on the way. What point on the river bank should he head for? [In other words locate a point $P \in \ell$ such that $AP + PB$ is a minimum. Assume that A is 3 miles from ℓ, B is 6 miles from ℓ, and the distance in miles between the foot of the perpendicular to ℓ from A and the foot of the perpendicular from B is 12.]

☆ 22. Of all the rectangles with perimeter p, what are the dimensions of the one which has the largest area-measure? [What are the dimensions of the one which has the smallest area-measure?]

☆ 23.



The rectangle DEFG is said to be inscribed in the triangle ABC. Locate the points D and E such that the area-measure of the inscribed rectangle is a maximum.

24. The perimeter of an isosceles triangle is 25 and the measure of one side is 10. What is the area-measure? [<u>Note</u>. Have you considered all cases?]

25. A diagonal of a quadrilateral and a segment whose end points belong to a pair of opposite sides of the quadrilateral bisect each other. Prove that the quadrilateral is a trapezoid.

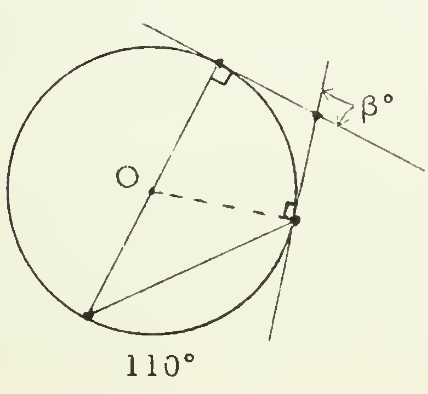
26. The measure of a chord $\overleftrightarrow{AB}$ of a circle of radius 3 is $3\sqrt{2}$. What is the length-measure of the major arc with end points A and B. [Repeat the problem if AB = 3.]

27. What is the sum of the measures of the angles of a convex decagon?

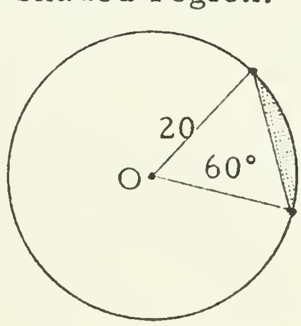
28. What is the measure of an exterior angle of a regular heptagon?

29. The sum of the measures of the angles of a convex polygon is 12 times the sum of the measures of the exterior angles [one at each vertex] of that polygon. How many sides does the polygon have?

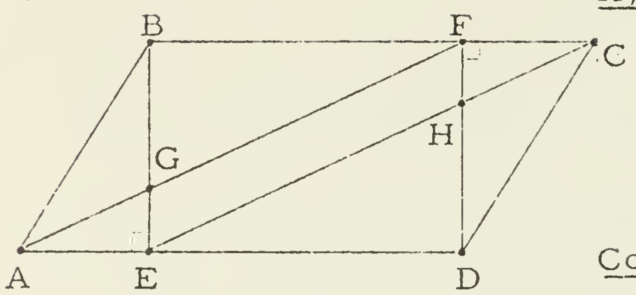
30. Find β.



31. Find the area-measure of the shaded region.



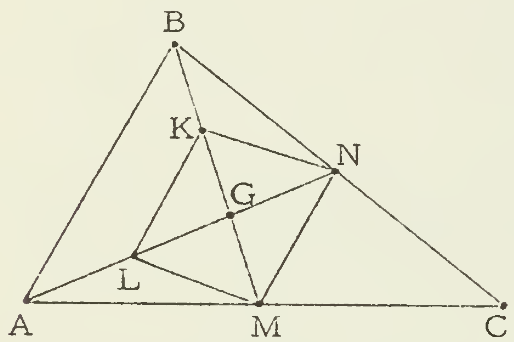
32.



Hypothesis: $\overline{BE}$ and $\overline{DF}$ are altitudes to the bases $\overline{AD}$ and $\overline{BC}$ of parallelogram ABCD

Conclusion: EHFG is a parallelogram

33.



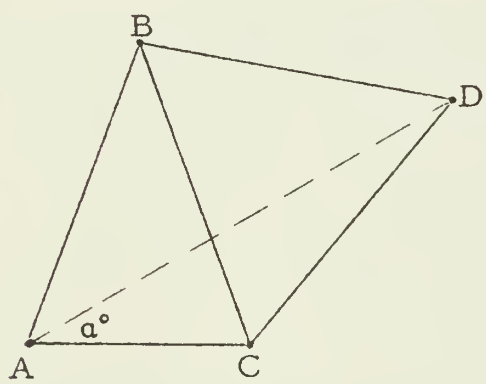
Hypothesis: $\overline{BM}$ and $\overline{MN}$ are medians which intersect in G,
K is the midpoint of $\overline{BG}$,
L is the midpoint of $\overline{AG}$

Conclusion: $\overline{KM}$ and $\overline{LN}$ bisect each other

34. Show that the centroid of an equilateral triangle is the circumcenter and the incenter.

35. Show that the area-measures of the six triangles formed by the three medians of a triangle are equal.

36.



Given: ΔABC is isosceles with $AB = BC$,
ΔBCD is equilateral

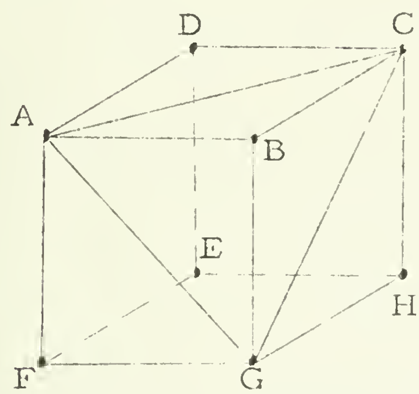
Find: a

37. P is a point in $\overline{AB}$ of parallelogram ABCD. Show that the area-measure of ΔPCD is the sum of the area-measures of ΔAPD and ΔPBC.

38. A pair of corresponding sides of two similar pentagons are 3 inches long and 12 inches long, respectively. Compare their perimeters. Compare their areas. If the area of the larger pentagon is 1600 square inches, what is the area of the smaller?

39. The lines which join a vertex of a parallelogram to the midpoints of the opposite sides trisect a diagonal.

40.

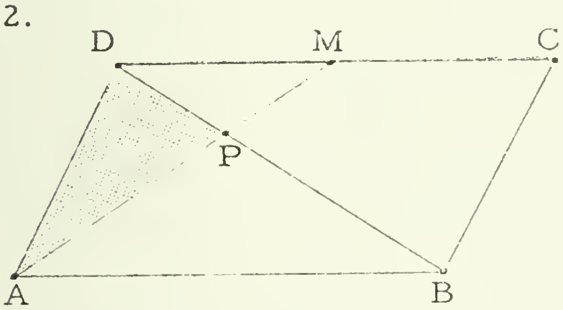


<u>Given:</u> Cube ABCD - EFGH, $\overleftrightarrow{AG}$, $\overleftrightarrow{CG}$, and $\overleftrightarrow{AC}$ are face diagonals, and each edge of the cube is 10 centimeters long

<u>Find:</u> Area of $\triangle ACG$

41. The diagonals of a rhombus have measures 6 and 8, respectively. What is the radius of the inscribed circle?

42.



<u>Given:</u> $DM = MC$, ABCD is a parallelogram

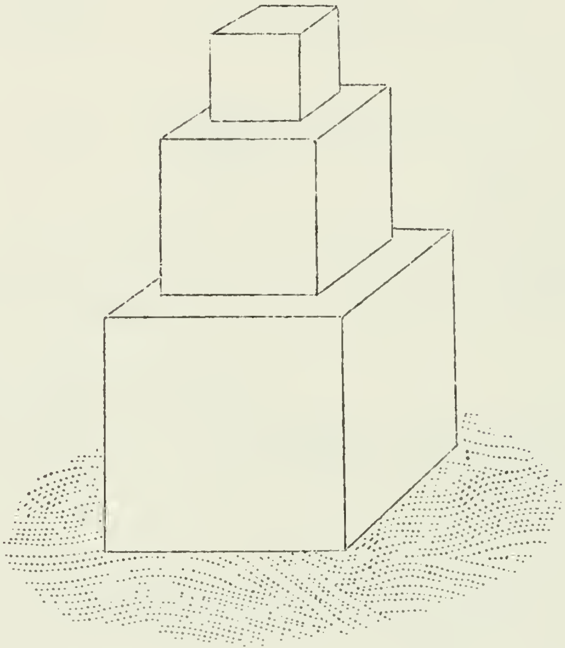
<u>Find:</u> The ratio of the area-measure of $\triangle APD$ to the area-measure of ABCD

43. A ship is sailing due north at a uniform rate. At 9:00 a.m. the bearing of a lighthouse is N 32° E. At 12 noon the lighthouse is due east and 15 miles away. Find the distance and bearing of the lighthouse at 1 p.m.

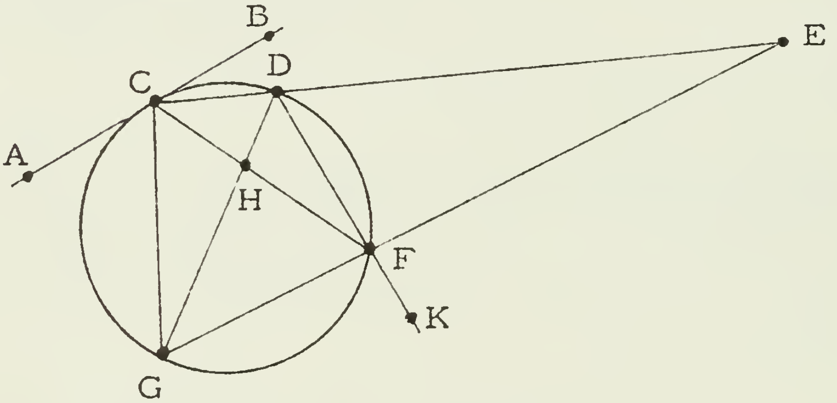
44. From a point 60 feet from the base of a building, the angle of elevation of the roof edge is an angle of 58°. Twelve feet below the roof edge a flagpole sticks out horizontally. From the door of the building, the angle of elevation of the end of the flagpole is an angle of 79°. What is the length of the flagpole?

45. The distance between the intersection of the diagonals of a rectangle and a side of measure 10 is 4. What is the area-measure of the rectangular region?

46. Three cubes have edges whose measures are consecutive whole numbers. The largest cube is placed on a table; the intermediate cube is placed on top of the largest cube; the smallest cube is placed on the top of the intermediate cube. Find the measures of the edges of the cubes if the area-measure of the exposed surface is 1604.



47. $\overleftrightarrow{AB}$ is a tangent at C, $\overleftrightarrow{CD}$ and $\overleftrightarrow{GF}$ are secants whose intersection is $\{E\}$, $\overleftrightarrow{DF}$ is a secant.



(a) Suppose $\overset{\frown}{CD}$ is an arc of $50°$, $\overset{\frown}{DF}$ is an arc of $x°$, $\overset{\frown}{FG}$ is and arc of $(x + 28)°$, and $\overset{\frown}{GC}$ is an arc of $(2x - 38)°$. Find the value of 'x' which meets these conditions. Then, use this value to find the degree-measures of $\overset{\frown}{DF}$, $\overset{\frown}{FG}$, and $\overset{\frown}{GC}$.

(b) Find the measures of $\angle CHG$, $\angle E$, $\angle ACG$, and $\angle GFK$.

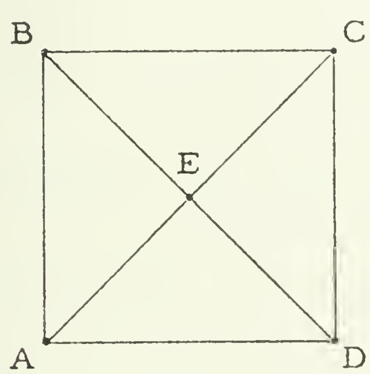
48. $\triangle ABC$ is inscribed in a circle, $\overset{\frown}{AB}$ is an arc of 68°, and $\angle BAC$ is an angle of 64°. If $\overleftrightarrow{BD}$ is a diameter which intersects $\overline{AC}$ in the point E, how many degrees are there in $\overset{\frown}{AD}$, $\overset{\frown}{BC}$, $\angle BEC$, and $\angle ABC$?

49. A highway engineer is planning a viaduct over some railroad tracks. The viaduct is to be 40 feet above the track bed and the approach is to have a 20% grade. How far from the tracks should the approach be started?

☆50. Suppose in quadrilateral ABCD, $\angle A \cong \angle C$ and $AB = CD$. Is quadrilateral ABCD a parallelogram? Justify your answer.

51. For what value of 'x' can '2x + 10' and '3x + 40' be the measures of two consecutive angles of a parallelogram.

52.



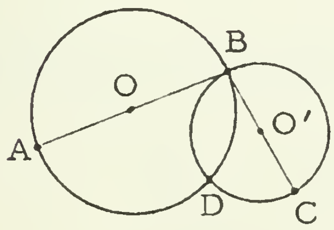
<u>Hypothesis:</u> $\overleftrightarrow{BD}$ and $\overleftrightarrow{AC}$ are perpendicular bisectors of each other, $\overline{AC} \cong \overline{BD}$

<u>Conclusion:</u> $\angle DAB$, $\angle ABC$, $\angle BCD$, and $\angle CDA$ are right angles

53. What is the length of the longest thin rod you could place in a box whose dimensions are 3 inches by 4 inches by 1 foot?

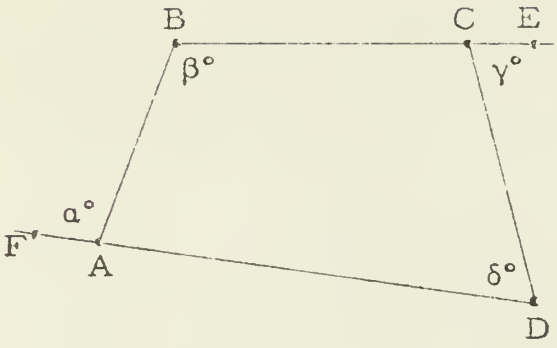
54. Explain how a man could "square" a sagging wooden gate by using only a scrap of wood, a steel rule, a hammer, a saw, and some nails.

55.



<u>Hypothesis:</u> B and D are two points of intersection of the circles whose centers are O and O′

<u>Conclusion:</u> A, D, and C are collinear

56. 

Hypothesis: ABCD is a convex quadrilateral, $\angle ECD$ and $\angle FAB$ are exterior angles

Conclusion: $\alpha + \gamma = \beta + \delta$

57. Consider the segment whose end points are the centers of the inscribed circle and an escribed circle [page 6-283] of a triangle. Prove that the midpoint of this segment belongs to the circumcircle of the triangle.

58. (a) Draw a scalene triangle $\triangle ABC$.

(b) Bisect $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{AC}$. Let the midpoints be M_1, M_2, and M_3, respectively.

(c) Draw the altitudes from A, B, and C of $\triangle ABC$. Let the feet of the altitudes be A_1, A_2, and A_3, respectively.

(d) Let the point of intersection of the altitudes be O.

(e) Bisect $\overleftrightarrow{OA}$, $\overleftrightarrow{OB}$, and $\overleftrightarrow{OC}$. Let the points of bisection be B_1, B_2, and B_3, respectively.

(f) Draw the circle which contains M_1, M_2, and M_3. Let the center be N.

☆(g) Prove that N is the midpoint of $\overline{B_2M_3}$.

☆(h) Prove that the circle which contains M_1, M_2, and M_3 also contains A_1, A_2, A_3, B_1, B_2, and B_3.

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Your work in Parts B and C on pages 6-91 and 6-92 established Theorem 3-3, which is:

> A point is equidistant from the two end points of a segment if and only if it belongs to the perpendicular bisector of the segment.

This theorem is often stated as:

> The perpendicular bisector of a segment is the locus [or: set] of points equidistant from the end points of the segment.

A <u>locus</u> is a set of points consisting of all points which satisfy a given condition. Actually, the word 'locus' is unnecessary since 'set of points' does just as well. However, the word 'locus' is a time-honored one, and you should become acquainted with it.

<u>Sample 1.</u> What is the locus of points at a distance 5 from the point P?

<u>Solution.</u> The locus is the circle with center P and radius 5.

<u>Sample 2.</u> What is the locus of points at a distance 5 from the line ℓ?

<u>Solution.</u> The locus is the union of two lines each parallel to ℓ and at a distance 5 from it.

This last sample illustrates the importance of considering <u>all</u> the points sstisfying the given condition. For example, one of the two parallel lines that make up the locus <u>is</u> a set of points satisfying the given condition. But, there are other points not on this line which satisfy the condition also.

In order to show that a geometric figure is a locus for a given condition, you must show that

(1) each point of the geometric figure satisfies the condition,

and that

(2) each point which satisfies the condition belongs to the geometric figure.

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59. Describe the locus of points which are

(a) at a distance 3 from each of two points;

(b) equidistant from two parallel lines;

(c) the midpoints of the radii of a circle;

(d) the midpoints of segments which are parallel to one side of a triangle and whose endpoints belong to the other sides;

(e) at a distance 2 from a segment of measure 7;

(f) the midpoints of congruent chords of a circle;

(g) the points of a circle <u>and</u> the vertices of inscribed angles congruent to a given inscribed angle;

(h) equidistant from the sides of an angle;

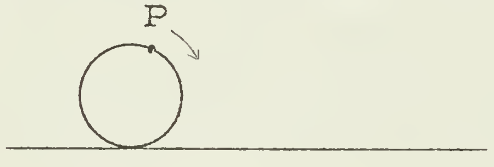
(i) equidistant from the sides of a triangle;

(j) the vertices of right triangles which have a common hypotenuse;

(k) the points of a circle and the vertices of inscribed angles which are supplementary to a given inscribed angle.

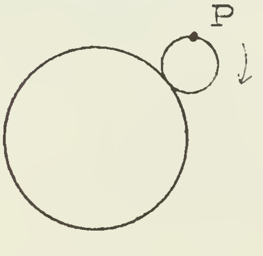
60. Sketch the locus of a point P on a circle as the circle

(a) "rolls along" a straight line;



(b) rolls along the inside of another circle;

(c) rolls along the outside of another circle.



61. Sketch the locus of points P which are

(a) equidistant from a point Q;

(b) equidistant from a point Q and a line ℓ;

(c) such that the sum of the distances between P and the points R and S is constant;

(d) such that the absolute value of the difference of the distances between the points R and S is constant.

62. Describe the locus in <u>space</u> of points which are

(a) equidistant from a point C;

(b) equidistant from a line ℓ;

(c) at a distance 1 from a circle of radius 5.